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SOLVING MOMENT PROBLEMS WITH APPLICATION TO STOCHASTICS

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SOLVING MOMENT PROBLEMS WITH APPLICATION TO STOCHASTICS

by

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1. Introduction

The theory of moments originated in the pioneer works of Shohat and Tamarkin (1943), Hoeffding (1955), Rogosinsky (1958), Ahiezer and Krein (1962), Karlin and Studden (1966), Kemperman (1968). It is also in the 1950s and '60s that moment theory became a separate mathematical discipline. Currently, it is now appropriate to talk about the moment problems as being a whole range of problems with applications to many mathematical theories.

The present paper is principally concerned with some applications of moment theory to probabilistic–statistical models. It is very difficult to encompass thoroughly all the problems and results linked to moment theory and stochastics. For some acquaintance with some problems of this nature we refer the reader to

(i) (Dependence of random quantities) Hoeffding (1955), Hoeffding and Shrikhande (1956);

(ii) (FKG–inequality, mass–transportation problems) Kemperman (1972), (1983), (1977), (1987), Haneveld (1985);

(iii) (Reliability theory, "aging" distributions) Basu and Simons (1983);

(iv) (Probability metrics) Sholpo (1983), Anastassiou (1987), Kuznezova–Sholpo and Rachev (1987), Rachev and Shortt (1989), Rachev (1991);

(v) (Limit theorems) Zolotarev (1986), Rachev (1991);

(vi) (Queueing theory) Anastassiou and Rachev (1989), (1990), and Rachev (1989).

The material presented in the first two sections provides results in the moment theory necessary for the study of different applied stochastic models.

The main results of Section 2 concern the explicit solutions of the following moment problems:

Moment problem 1: *Find*

$$(1.1) \qquad \sup_{\mu} \int_S |x-y|^p \, \mu(dx, dy), \; S \subset R^2, \; p \geq 1$$

and

$$(1.2) \qquad \inf_{\mu} \int_S |x-y|^p \, \mu(dx, dy), \; S \subset R^2,$$

where the supremum (resp. *infimum*) *is taken over the set of all probability measures* μ *with support on* S *having fixed marginal moments*

$$(1.3) \qquad \int_S x \, \mu(dx, dy) = \alpha, \; \int_S y \, \mu(dx, dy) = \beta.$$

Problem (1.2) with fixed marginal distributions

$$(1.4) \qquad \mu(\cdot, S) = \mu_1(\cdot), \; \mu(S, \cdot) = \mu_2(\cdot)$$

is the well-known Kantorovich problem on mass transportation (see Kemperman (1983), Rachev (1984), (1991)). Extensions of the moment problems (1.1), (1.2), (1.3) for random processes will be carried out in Section 4.

Moment problem 2: *For given* $x_0 \in R$ *and positive* α *find the Kantorovich radius*

$$(1.5) \qquad \sup \; E \mid X - x_0 \mid^{\alpha},$$

where the supremum is over all random variables X *with fixed moments* $EX = p$ *and* $EX^2 = q$.

The solution of problem 2 will lead to best approximation of a "real" queueing systems (govern by a random mechanism) by means of "ideal" deterministic queueing models commonly applied in practice.

The concern of section 3 is with moment problems relative to methods of rounding and apportionment (see Mosteller, Youtz and Zahn (1967), Diaconis and Freedman (1979) and Balinski and Young (1982)). For any non–negative x define the *c–rounding* $(0 \le c \le 1)$:

$$[x]_c = \begin{cases} m, & \text{if } m \le x \le m + c \ , \\ m + 1, & \text{if } m + c < x \le m + 1. \end{cases}$$

In the apportionment theory $c = 0$ corresponds to the Adams method; $c = 1/$ corresponds to the Webster method (or conventional rounding, Mosteller–Youtz–Zahn "broken stick" rule of rounding); $c = 1$ correspond to Jefferson method. The objectives of section 3 are the following

Moment problem 3: *Find*

$$(1.6) \qquad \sup \int_A [t]_c \, \mu(dt), \ A = [0, a] \text{ or } [0, \infty)$$

and

$$(1.7) \qquad \inf \int_A [t]_c \, \mu(dt), \ A = [0, a] \text{ or } [0, \infty)$$

over the set of all probability measures with support A having fixed rth moment

$$(1.8) \qquad \int_A t^r \mu(dt) = d_r,\ r > 0,\ d_r > 0 .$$

Moment problem 4: *Find* (1.6) *and* (1.7) *subject to* (1.8) *and*

$$(1.9) \qquad \int_A t\ \mu(dt)) = d_1 .$$

Later in section 6 we shall return to the solutions of problems 3 and 4 in order to derive the "best" rule of c–roundings.

Section 4 deals with infinite–dimensional analogs of moment problems 1 and 2. Namely let $C[0,1]$ be the space of all continuous functions on $[0,1]$ with supremum norm $\|x\| = \sup\{|x(t)|: 0 \le t \le 1\}$. Consider the space $\mathcal{S}(C[0,1])$ of all random variables (r.v.'s) on a nonatomic probability space $(\Omega, \mathcal{A}, Pr)$. Let $\mathcal{M}$ be the class of all strictly increasing continous functions $f:[0,\infty] \to [0,\infty]$, $f(0) = 0$, $f(\infty) = \infty$.

Let T be a set of finite points

$$(1.10) \qquad 0 \le t_1 < t_2 < \ldots \le t_N \le 1 .$$

Consider

Moment problem 5: *Given* $h, g_i \in \mathcal{M}$ $(i = 1, \ldots, N)$

find

(1.11) $\inf \mathbf{E}\, h(\|X - Y\|)$

over the set of all possible joint distributions of X *and* Y *subject to the moment constraints:*

(1.12) $\mathbf{E}\, g_i(|X(t_i)|) = a_i$, $\mathbf{E}\, g_i(|Y(t_i)| = b_i$.

In other words, having observations of two random processes (more precisely, we suppose the moments (1.12) are known), we want to evaluate the minimal possible distance $\mathbf{E}h(\|X - Y\|)$ between the processes X and Y. We shall determine the minimum in (1.11) and show that essentially for any ν greater that this minimum there exist X and Y enjoying (1.12) and $\mathbf{E}h\{\|X - Y\|) = \nu$.

Section 5 discusses applications of moment problems 1 to the problem of best approximation of a queueing system with known moment characteristics. Namely, suppose our "real" queueing system is of type $G|G|1|\infty$ (for some acquaintance with the usual notations in the queuing theory we refer to Borovkov (1984), Kalashnikov and Rachev (1990)). For this system the sequences of nonnegative r.v.'s (possibly dependent and nonidentically distributed) $\bar{e} = \{e_n\}_{n \in N}$, $\bar{s} = \{s_n\}_{n \in N}$ $(N = 1,2,...)$ are viewed as the sequences of interarrival and service times. Looking at $\bar{e}$ and $\bar{s}$ as "input" flows we define (as an "output" flow) the sequence of waiting times

(1.13) $w_1 = 0$, $w_{n+1} = (w_n + s_n - e_n)_+$, $n \in N$,

where $(\cdot)_+ = \max(0, \cdot)$. Since the distribution of $\bar{w} = \{w_n\}_{n \in N}$ is not known, the aim is to approximate, model, or simulate the "real" system determined by the triplet

$(\bar{e}, \bar{s}, \bar{w})$ with a "simpler" queueing model $(\bar{e}^*, \bar{s}^*, \bar{w}^*)$. Assuming that the marginal distributions (the laws of e_i, s_i) are known, Borovkov (1984) (Chapter 4) and Kalashnikov and Rachev (1990) examined different approximating models $(\bar{e}^*, s^*, \bar{w}^*)$ and estimated the possible discrepancy between the "real" system $(\bar{e}, \bar{s}, \bar{w})$ and the "ideal" model $(\bar{e}^*, \bar{s}^*, \bar{w}^*)$. In section 5 we shall relax the constraints "the laws of e_i's and s_i's are known" by "certain moment characteristics of e_i's and s_i's are fixed". In this setup the solutions of moment problems 1 are used in cases when the "ideal" model is not deterministic, say $G|G|1|\infty$, but one with simpler structure. We invoke moment problem 2 when the approximating model has some deterministic components, like $D|G|1|\infty$ (i.e. e_j^*'s are constants), or $D|D|1|\infty$ (i.e. e_j^*'s and s_j^*'s are constants).

The interplay between the moment problems 3 and 4 and the problem of rounding of random proportions is discussed in section 6. Given a vector $x = (X_1,...X_n)$ of r.v.'s consider the sum $X_1 + ... + X_n$. If X_i's are uniformly distributed on the simplex $\{s_i \geq 0, s_1 + ... + s_n = 1\}$, X_i's can be treated as proportions and clearly $S_n := X_1 +...+ X_n = 1$. In the later case, Mosteller, Youtz and Zahn (1967) and Diaconis and Freedman (1979) estimated the probability that $S_n = S_n^*$ when S_n^* is the sum of conventional roundings $[X_1]_{1/2} + ... + [X_n]_{1/2}$. In section 6 we examine the closeness between S_n and S_n^* in the case of i.i.d. observations X_i with *only* known one or two moments. The results are based on the solutions of moment problems 3 and 4 and the theory of probability metrics (cf. Zolotarev (1986) and Rachev (1991)).

2. Moment problems of Kantorovich type and Kantorovich radius.

This section deals with moment problems 1 and 2. We start with problem (1.1) –

(1.3). Let $S = [a, b] \times [c, d] \subset \mathbb{R}^2$ and $\varphi(x_1, x_2) = |x_1 - x_2|^p$, $p \geq 1$. Suppose $(\alpha, \beta) \in S$ and denote by $U = U(\varphi, \alpha, \beta)$ the supremum in (1.1) subject to

$$(2.1) \qquad \int_S x_1 \, \mu(dx_1, dx_2) = \alpha, \quad \int_S x_2 \, \mu(dx_1, dx_2) = \beta.$$

Theorem 2.1

$$(2.2) \qquad U = D\delta + T,$$

where

$$D := [|b-d|^p + |a-c|^p] - [|b-c|^p + |a-d|^p]$$

$$T := (1-B)\,|b-c|^p + (B+C-1)\,|a-c|^p + (1-C)\,|a-d|^p,$$

$$\left[B := \frac{b-\alpha}{b-a}, \; C := \frac{d-\beta}{d-c}\right]$$

$$\delta := \max(0, 1-B-C).$$

Proof. From the convexity φ, a simple checking shows that $D \leq 0$. From Proposition 1.4.6 and Lemma 1.4.7 (Alfsen (1971)) the optimal measure in (1.1) is carried by the four extreme points of S. From the given moment conditions one can determine the masses of three of the verteses of S as a function of ζ, the mass of the fourth vertex of S. Optimization w.r.t. $\zeta \in (0, 1)$ of $\int_{ext(S)} \varphi \, d\mu$ gives (2.2). □

Remark. Since φ is convex on any $\underline{O} \subset \mathbb{R}^2$ then given (2.1)

$$\inf_{\mu\ \underline{O}} \int \varphi \, d\mu = |\alpha - \beta|^p .$$

Next, we extend Theorem 2.1 to nonbounded regions S . Namely, consider the following stripes in $\mathbb{R}^2$: for $b > 0$,

$$S_1^b := \{(x, y) : y = x + b' \text{ , where } 0 \le b' \le b\} ,$$

$$S_2^b := \{(x, y) : y = x - b' \text{ , where } 0 \le b' \le b\} ,$$

$$S^b := S_1^b \cup S_2^b .$$

Theorem 2.2. Assume that $0 < p \le 1$

(i) If $S = S_1^b$ or S_2^b , $(\alpha, \beta) \in S$ *then*

$$U := U(\varphi , \alpha , \beta) = |\alpha - \beta|^p$$

where U *is given by* (1.1) *and* (2.1).

(ii) *If* $S = S^b$ *then* $U = b^p$.

(iii) *If* L *is the lower bound* (1.2) *subject to (2.1) and if* $S = S_1^b$ *or* $S = S_2^b$ or $S = S^b$, $(\alpha, \beta) \in S$ then

$$L := L(\varphi ; \alpha , \beta) = b^{p-1} |\alpha - \beta| .$$

Proof. We follow the Kemperman (1968) geometric approach to solve these moment problems.

(i) It is true because of concavity of $\varphi(x_1, x_2) = |x_1 - x_2|^p$, $0 < p \leq 1$ in S.

(ii) Here the upper envelope of the convex hull of $(x_1, x_2, |x_1 - x_2|^p)$, $(x_1, x_2) \in S^b$ is a plane stripe made out of the lines

$$(l_1): \quad \left.\begin{aligned} x_1 &= t \\ x_1 &= b + t \\ x_3 &= b^p \end{aligned}\right\} \quad \text{and } (l_2): \quad \left.\begin{aligned} x_1 &= t \\ x_2 &= -b + t \\ x_3 &= b^p \end{aligned}\right\}$$

The plane through (l_1), (l_2) has equation $x_3 = b^p$.

Therefore

$$U = \sup_{\mu} \int_{S^b} |x_1 - x_2|^p \, \mu(dx_1, dx_2) = b^p,$$

independently of α, β.

(iii) We consider the case of $S = S_1^b$, the cases of $S = S_2^b$ or $S = S^b$ are treated similarly.

Here the lower envelope of the convex hull of $(x_1, x_2, |x_1 - x_2|^p)$, $(x_1, x_2) \in S_1^b$ is a plane stripe generated by the lines

$$(l): \quad \left.\begin{aligned} x_1 &= t \\ x_2 &= t \\ x_3 &= 0 \end{aligned}\right\} \quad \text{and } (l_1): \quad \left.\begin{aligned} x_1 &= t \\ x_2 &= b + t \\ x_3 &= b^p \end{aligned}\right\}$$

The plane through (l) and (l_1) has equation $x_3 = b^{p-1}(x_2 - x_1)$.

Therefore

$$L = \inf_{\mu} \int_{S_1^b} |x_1 - x_2|^p \, \mu(dx_1, dx_2) = b^{p-1} (\beta - \alpha) . \qquad \square$$

In the following theorem we consider the stripes in $\mathbb{R}^2$: for $b, \gamma > 0$,

$$S_1^b := \{(x, y): y = x + b', \text{ where } 0 \le b' \le b\},$$

$$S_2^\gamma := \{(x, y): y = x - \gamma', \text{ where } 0 \le \gamma' \le \gamma\},$$

$$S^{b,\gamma} = S_1^b \cup S_2^\gamma .$$

Theorem 2.3 Assume that $p \ge 1$.

(i) If $S = S_1^b$, $(\alpha, \beta) \in S$ then

$U := U(\varphi, \alpha, \beta) = b^{p-1}(\beta - \alpha)$

where U is given by (1.1) and (2.1).

(ii) If $S = S_2^\gamma$, $(\alpha, \beta) \in S$ then

$U := U(\varphi, \alpha, \beta) = \gamma^{p-1}(\alpha - \beta)$.

(iii) If $S = S^{b,\gamma}$, $(\alpha, \beta) \in S$ then

$$U = \frac{(b^p - \gamma^p) \cdot (\beta - \alpha - b) + b^p (b + \gamma)}{b + \gamma} .$$

Proof. Note here that $\varphi(x_1, x_2) = |x_1 - x_2|^p$ is convex for $p \ge 1$. Otherwise the proof is similar to the proof of Theorem 2.2. $\square$

Next, consider moment problem 2, i.e. for given $x_0 \in \mathbb{R}$, $\alpha > 0$, $p \in \mathbb{R}$, $q > 0$ $(p^2 \le q)$, $-\infty \le a < b \le +\infty$ find the *Kantorovich radius*

$$K := K(x_0; \alpha, p, q, a, b) \tag{2.10}$$

$$:= \sup\{E\,|X - x_0|^\alpha : X \in [a, b] \text{ a.s., } E\,X = p, E\,X^2 = q\} .$$

Theorem 2.4. *(Case (A): $\alpha \geq 2$, $-\infty < a < b < +\infty$)*

Suppose X's in (2.10) have a compact support $[a, b]$. *Let* $x_0 = (a + b)/2$, $a \leq p \leq b$ *and* $0 \leq q \leq b^2 + (a + b)(p - b)$. *Then*

$$K \leq \left(\frac{b - a}{2}\right)^{\alpha-2} \left[q - p(a + b) + \left(\frac{a + b}{4}\right)^2\right]$$

Moreover, if there exist $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 \leq 1$ *such that*

$$p = \left(\frac{a + b}{2}\right) + \left(\frac{b - a}{2}\right)(\lambda_1 - \lambda_2) \text{ and } q = \frac{(a + b)^2}{4} + \left(\frac{b^2 - a^2}{2}\right)(\lambda_1 - \lambda_2)$$

$$+ \frac{(b - a)^2}{4}(\lambda_1 + \lambda_2)$$

then

$$K = \left(\frac{b - a}{2}\right)^{\alpha-2} \left[q - p(a + b) + \left(\frac{a + b}{4}\right)^2\right].$$

Proof. By calling $Y := X - x_0$ we have $E(Y) = P$ and $E(Y^2) = Q$, where $P := p - x_0$ and $Q := q - 2px_0 + x_0^2$. Equivalently, we would like to find $K = K(Y, 0) = \sup_Y E(|Y|^\alpha)$, $\alpha \geq 2$ such that $EY = P$ and $EY^2 = Q$. To calculate K we use standard geometric moment thoery from Kemperman (1968). Here note that the graph of $(t^2, |t|^\alpha) = (z, z^{\alpha/2})$, $z \geq 0$. Also call $A := x_0 - a = b - x_0 > 0$, thus $|x - x_0| \leq A$, for all $x \in [a, b]$.

Let $\mathcal{O} := (0, 0, 0)$, $\mathcal{A} := (A, A^2, 0)$, $\mathcal{B} = (-A, A^2, 0)$. Then $(P, Q) \in \mathcal{O}\overset{\Delta}{\mathcal{A}}\mathcal{B}$ iff there exist $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 \leq 1$ with $P = A \cdot (\lambda_1 - \lambda_2)$ and $Q = A^2 \cdot (\lambda_1 + \lambda_2)$ iff there exist $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 \leq 1$ with

$$p = \left[\frac{a + b}{2}\right] + \left[\frac{b - a}{2}\right] \cdot (\lambda_1 - \lambda_2) \text{ and}$$

$$q = \frac{(a + b)^2}{4} + \left[\frac{b^2 - a^2}{2}\right] \cdot (\lambda_1 - \lambda_2) + \frac{(b - a)^2}{4} \cdot (\lambda_1 + \lambda_2).$$

When $(P, Q) \in \mathcal{O}\overset{\Delta}{\mathcal{A}}\mathcal{B}$ then $P^2 \leq Q$. According to moment theory, there exist r.v.'s Y such that $EY = P$, $EY^2 = Q$ iff the moment point (P, Q) is in the interior or boundary of the parabola (t, t^2) i.e. when $P^2 \leq Q$, $-A \leq P \leq A$ and $0 \leq Q \leq A^2$.

Equivalently, there exist r.v.'s X such that $EX = p$, $EX^2 = q$ iff $p^2 \leq q$, $a \leq p \leq b$ and $0 \leq q \leq b^2 + (a + b) \cdot (p - b)$. The upper part of the convex hull of $(t, t^2, |t|^\alpha)$, $\alpha \geq 2$, $-A \leq t \leq A$, which is above the triangle $\mathcal{O}\overset{\Delta}{\mathcal{A}}\mathcal{B}$ is the triangle $\{(0, 0, 0), (A, A^2, A^\alpha), (-A, A^2, A^\alpha)\}$. The plane through the last triangle has equation $z = A^{\alpha-2}y$. Hence, $K = z$, where (P, Q, z) belongs to this plane. Therefore

$K = A^{\alpha-2} Q$, $(A = \frac{b - a}{2}$ and $Q = q - p(a + b) + \frac{(a + b)^2}{4})$. Next one can observe that the curve $(t, t^2, |t|^\alpha)$ is below the plane $z = A^{\alpha-2}y$, for all $-A \leq t \leq A$. Therefore the upper part of the convex hull of $(t, t^2, |t|^\alpha)$ is below or on that plane. Consequently,

$$K \leq A^{\alpha-2} Q$$

for all (P, Q) in or on the parabola (t, t^2) with $P^2 \leq Q$, $-A \leq P \leq A$ and $0 \leq Q \leq A^2$. □

Theorem 2.5. (*Case (B):* $0 < \alpha \le 2$, $a = -\infty$, $b = +\infty$). For any $x_0 \in \mathbb{R}$, $p \in \mathbb{R}$, $q > 0$, $p^2 \le q$, the Kantorovich radius K admits the following value

$$K = K(x_0 ; \alpha ; p; q) = (q - 2x_0p + x_0^2)^{\alpha/2} .$$

Proof. Case B1 : $1 < \alpha \le 2$ (Case B2 : $0 < \alpha \le 1$ as similar is omitted). Call $Y := X - x_0$, thus $EY = P$, $EY^2 = Q$, where $P := p - x_0$, $Q := q - 2x_0p + x_0^2$. We would like to find $K = \sup_Y E|Y|^\alpha$, $1 < \alpha \le 2$ such that $EY = P$, $EY^2 = Q$. Here observe that $P^2 \le Q$ iff $p^2 \le q$.

Let $V := \operatorname{conv}_{t \in \mathbb{R}} (t, t^2, |t|^\alpha)$ and S be its upper part. This is formed by the line segments parallel to x-axis touching the curve $(t, t^2, |t|^\alpha)$. The associated moment point on (xy)-plane is (P, Q, o). Thus, $K = (\sqrt{Q})^\alpha$, $1 < \alpha \le 2$. □

Theorem 2.6. (*Case (C):* $0 < \alpha \le 2$, $-\infty < a < b < +\infty$).

For any $x_0 \in (a, b)$; $p \in \mathbb{R}$, $p^2 \le q$,

$$K = \sup \{E\,|X - x_0|^\alpha : X \in [a, b] \text{ a.s., } EX = p,\ EX^2 = q\} .$$

Set $P = p - x_0$, $Q = q - 2x_0p + x_0^2$, $A(x_0) = a - x_0$, $B(x_0) = b - x_0$, $C(x_0) = \min(-A(x_0), B(x_0))$.

(i) *Suppose* $0 \le Q \le C^2(x_0)$ *then* $K = Q^{\alpha/2}$.

(ii) *Suppose* $Q \ge C^2(x_0)$ *and* $(A(x_0) + B(x_0))P - Q - A(x_0)B(x_0) \ge 0$.

Then $K \le Q^{\alpha/2}$

Proof: (i) Case : $1 < \alpha \leq 2$. ((ii) Case : $0 < \alpha \leq 1$ as similar is omitted). Call $Y := X - x_0$, then $E(Y) = P$, $EY^2 = Q$.

Hence, we would like to find $K = \sup_Y E(|Y|^\alpha)$, $1 < \alpha \leq 2$ given that $EY = P$, $EY^2 = Q$. Note that $P^2 \leq Q$ iff $p^2 \leq q$.

(i) Suppose $0 \leq Q \leq C^2(X_0)$, then $|P| \leq C(x_0)$. Call $V = \operatorname{conv}_{A(x_0) \leq t \leq B(x_0)} (t, t^2, |t|^\alpha)$, $1 < \alpha \leq 2$.

Let S be the upper part of the convex hull V when $-C(x_0) \leq t \leq C(x_0)$. This is formed by line segments parallel to the x–axis touching $(t, t^2, |t|^\alpha)$. Our moment point on (xy)–plane is (P, Q, 0) with $0 \leq Q \leq C^2(x_0)$. Thus $K = (\sqrt{Q})^\alpha$, $1 < \alpha \leq 2$.

(ii) Suppose $Q \geq C^2(x_0)$ and $(A(x_0) + B(x_0))$. $P - Q - A(x_0) \cdot B(x_0) \geq 0$.

Hence, the moment point (P, Q) is below the line

(l) : $y = (A(x_0) + B(x_0)) \cdot x - A(x_0) \cdot B(x_0)$.

Therefore $K \leq (\sqrt{Q})^\alpha$.

Theorem 2.7. (*Case (D):* $1 \leq \alpha \leq 2$, $-\infty < a < b \leq \infty$) .

For any $p \in \mathbb{R}$, $p^2 \leq q$, $a \leq p \leq b$, *set* $P = p - a$, $Q = q - 2ap + a^2$, $B = b - a$. *Suppose* $Q \leq BP$. *Then*

$$K := \sup\{E\,|X - a|^\alpha : X \in [a, b],\ E\,X = p\ ,\ E\,X^2 = q\}$$

$$= P^{2-\alpha} Q^{\alpha - 1} .$$

Proof. Call $Y = X - a$, then $E(Y) = P$, $EY^2 = Q$. Thus we want to find $K = \sup_Y E(|Y|^\alpha)$, $1 \le \alpha \le 2$ given that $EY = P$, $EY^2 = Q$. Call $t := x - a$, then $0 \le t \le B$ when $a \le x \le b$. Note that $P^2 \le Q$ iff $p^2 \le q$, also $0 \le P \le B$ iff $a \le p \le b$. From $Q \le B \cdot P$ we get $Q \le B^2$. Call $V = \operatorname{conv}_{0 \le t \le B} (t, t^2, t^\alpha)$, $1 \le \alpha \le 2$. Let S be the upper part of the convex hull V. This is formed by all line segments $\overline{[0, 0, 0]}$, $\overline{[t, t^2, t^\alpha]}$, $0 \le t \le B$. The line through (0, 0) and (B, B^2) is $y = B \cdot x$. Since $Q \le B \cdot P$, (P, Q) is an admissible moment point. Call $0 := (0, 0, 0)$, $A := (P, Q, 0)$, $M := (P, Q, z) \in S$, $\mathcal{K} := (x_0, x_0^2, 0)$; $x_0 := \frac{Q}{P}$, $\mathcal{L} := (x_0, x_0^2, x_0^\alpha) \in S$, such that O, M, $\mathcal{L}$ are on the same line. Obviously, the triangle OAM and $\mathcal{O}\mathcal{K}\mathcal{L}$ are similar, therefore $\frac{\overline{OA}}{\overline{O\mathcal{K}}} = \frac{\overline{AM}}{\overline{\mathcal{K}\mathcal{L}}}$ and $\overline{AM} = \frac{\overline{OA}}{\overline{O\mathcal{K}}} \cdot \overline{\mathcal{K}\mathcal{L}}$. Here $\overline{AM} = z$, $\overline{OA} = \sqrt{P^2 + Q^2}$, $\overline{O\mathcal{K}} = \frac{Q}{P^2} \cdot \sqrt{P^2 + Q^2}$ and $\overline{\mathcal{K}\mathcal{L}} = (\frac{Q}{P})^\alpha$. Hence $z = P^{2-\alpha} \cdot Q^{\alpha-1}$ and $K = z$. □

Theorem 2.8. *(Case (E):* $1 \le \alpha \le 2$, $-\infty \le a < b < \infty$*)*

For any $p \in \mathbb{R}$, $p^2 \le q$, $a \le p \le b$, *set* $P = p - b$, $Q = q - 2bp + b^2$, $\theta = a - b$. *Suppose* $Q \le \theta P$. *Then*

$$K := \sup\{|X - b|^\alpha : X \in [a, b], E X = p, E X^2 = q\}$$

$$= |P|^{2-\alpha} Q^{\alpha-1}.$$

Proof. As similar to Theorem 2.7 is omitted. □

3. Moment Problems Related to Rounding Proportions.

The objective in this section is the solution of moment problems 3 and 4, see (1.6) and (1.9). We first recall the definition of c-*rounding* $(0 \le c \le 1)$: for any $x \ge 0$,

$$[x]_c = \begin{cases} m \ , & \text{if } m \le x < m + c \\ m+1 \ , & \text{if } m + c \le x < m + 1 \ , \end{cases} \quad (m \in N \cup \{0\})$$

or equivalently,

$$[X]_c = \begin{cases} 0, & \text{if } 0 \le x \le c \\ m, & \text{if } m - 1 + c < x \le m + c, \quad (m \in N) \ . \end{cases}$$

The next four theorems deal with problem 3. In the sequel, the underlined probability space is assumed to be nonatomic and thus the space of laws of nonnegative r.v.'s coincides with the space of all Borel probability measures on R_+ .

Theorem 3.1. *Let* $c \in (0, 1)$, $r > 0$, $0 < a < \infty$, $d > 0$, *and*

$$U := U_{[\cdot]_c}(a, r, d) := \sup\{E\,[X]_c : 0 \le X \le a \ \text{a.s.}, \{E\,X^r\}^{1/r} = d\} \ .$$

Set $n = [a]$.

(I) *If* $n + c < a$, $n + c \le d \le a$, *then* $U = n + 1$.

(II) *If* $n + c \ge a$, $n - 1 + c \le d \le a$, *then* $U = n$.

(III) *If* $0 < a \le c$, *then* $U = 0$.

(IV) *If* $0 < r \le \frac{\ln 2}{\ln(1 + 1/c)} (< 1)$, $n + c < a$

and $0 \le d \le n + c$, *then* $U = (n + 1)\, d^r\, (n + c)^{-r}$.

(V) *If* $0 < r \le \frac{\ln 2}{\ln(1 + 1/c)}$, $n + c \ge a$ *and* $0 \le d \le n - 1 + c$, *then*

$$U = n d^r (n - 1 + c)^{-r}.$$

(VI) *If* $r \ge 1$ *and* $0 \le d \le c$, *then* $U = d^r c^{-r}$.

(VII) *Suppose* $r \ge 1$. *If either*

(a) $n + c < a$ *and determine* $k \in \{1, ..., n\}$ *by* $k - 1 + c \le d < k + c$, *or*

(b) $n + c \ge a$ *and determine* $k \in \{1, ..., n - 1\}$ *by* $k - 1 + c \le d < k + c$. *Then*

$$U = k + \frac{d^r - (k - 1 + c)^r}{(k + c)^r - (k - 1 + c)^r} \le 1 - c + d.$$

Proof. It follows that $U = \Psi(d^r)$, where $A_1 = \{(u, \Psi(u)) : 0 \le u \le a^r\}$ describes the upper boundary of the convex null conv A_0 of the curve $A_0 = \{(t^r, [t]_c) : 0 \le t \le a\}$. Note that A_0 consists of the following parts:

Case (I) Assume that $n + c < a$.

(i) The closed interval $[0, c]$.

(ii) For $k = 1, ..., n$ the half open horizontal line segments $(P_k, Q_k]$, where $P_k = ((k - 1 + c)^r, k)$ and $Q_k = ((k + c)^r, k)$.

(iii) The half open non–empty horizontal line segment $(P_{n+1}, Q_*]$, where $P_{n+1} =$

$((n + c)^r, n + 1)$ and $Q_* = (a^r, n + 1)$.

Case (II) Assume that $n + c \geq a$.

(i) The closed interval $[0, c]$.

(ii) For $k = 1, \ldots, n - 1$ the half open horizontal line segments $(P_k, Q_k]$, where $P_k = ((k - 1 + c)^r, k)$ and $Q_k = ((k + c)^r, k)$.

(iii) The half open non–empty horizontal line segment $(P_n, Q_*]$, where $P_n = ((n - 1 + c)^r, n)$ and $Q_* = (a^r, n)$. Parts (iii) of cases (I), (II) are parts of the upper boundary A_1 of conv A_0, yielding $U = n + 1$; $U = n$, respectively. Here we need $n + c \leq d \leq a$; $n - 1 + c \leq d \leq a$, respectively. The line segment from P_k to P_{k+1} has slope $\dfrac{1}{(k + c)^r - (k - 1 + c)^r}$ which is decreasing in k when $r \geq 1$ and increasing in k when $0 < r \leq 1$. Call $P_0 = (0, 0)$. Slope $(\overline{P_0P_1}) = \dfrac{1}{c^r}$.

Observe that slope $(\overline{P_0P_1}) \geq$ slope $(\overline{P_1P_2})$ for $r \geq 1$. When $0 < r \leq \dfrac{\ln 2}{\ln(1 + c^{-1})}$ (< 1) we have slope $(\overline{P_0\ P_1}) \leq$ slope $(\overline{P_1\ P_2})$. Consequently, if $0 < r \leq \dfrac{\ln 2}{\ln(1 + c^{-1})}$ then A_1 consists of the following two line segments:

Case (I) of $n + c < a$

$(\overline{P_0P_{n+1}})$, $(\overline{P_{n-1}\ Q_*})$, where $P_0 = (0, 0)$, $P_{n+1} = ((n + c)^r, n + 1)$, $Q_* = (a^r, n + 1)$. Here $0 \leq d \leq n + c$. Thus, $\dfrac{U}{n + 1} = \dfrac{d^r}{(n + c)^r}$, giving us

$$U = (n + 1)\frac{d^r}{(n + c)^r}.$$

Case (II) of $n + c \geq a$

$(\overline{P_0P_n})$, $(\overline{P_nQ_*})$, where $P_n = ((n - 1 + c)^r, n)$ and $Q_* = (a^r, n)$. Here $0 \leq d \leq n -$

$1 + c$. Thus $\frac{U}{n} = \frac{d^r}{(n - 1 + c)^r}$ i.e. $U = \frac{n \cdot d^r}{(n - 1 + c)^r}$.

On the other hand, if $r \geq 1$, then A_i is composed of the following line segments:

Case (I) of $n + c < a$

$(\overline{P_k P_{k+1}})$, $k = 0, 1, \ldots, n$ and $(\overline{P_{n+1} Q_*})$, where $P_k = ((k - 1 + c)^r, k)$, $Q_* = (a^r, n + 1)$.

Case (II) of $n + c \geq a$

$(\overline{P_k P_{k+1}})$, $k = 0, 1, \ldots, n - 1$ and $(\overline{P_n Q_*})$, where $P_k = ((k - 1 + c)^r, k)$, $Q_* = (a^r, n)$.

In either case (I), (II), when $0 \leq d \leq c$ we get $U = d^r c^{-r}$. Note that slope $\overline{P_k P_{k+1}} = \frac{1}{(k+c)^r - (k-1+c)^r}$ which is decreasing in k, $r \geq 1$. The equation of the line through P_k, P_{k+1} has as follows

$$y - k = \frac{(x - (k-1+c)^r)}{\left[(k+c)^r - (k-1+c)^r\right]}.$$

Here $k - 1 + c \leq d < k + c$. Therefore

$$U = k + \left[\frac{d^r - (k-1+c)^r}{(k+c)^r - (k-1+c)^r}\right],$$ where in case (I), $k \in \{1, \ldots, n\}$ and in case (II), $k \in \{1, \ldots, n-1\}$. By convexity of x^r, $r \geq 1$; $x \geq 0$ wee obtain

$$U \leq (1 - c) + d. \qquad \square$$

We extend Theorem 3.1 in the case $a = +\infty$ as follows.

Theorem 3.2 *Let* $0 < c < 1$, $r > 0$, $d > 0$ *and*

$$U := U_{[\cdot]_c}(r, d) := \sup\{E\,[X]_c : X \geq 0 \text{ a.s.}, \{E\,X^r\}^{1/r} = d\}.$$

(I) *If* $0 < r < 1$, *then* $U = +\infty$.

(II) *If* $r \geq 1$ *and* $0 \leq d \leq c$, *then* $U = d^r c^{-r}$.

(III) *Suppose* $r \geq 1$. *Define* $k \in N$ *by* $k - 1 + c \leq d < k + c$. *Then*

$$U = k + \frac{d^r - (k - 1 + c)^r}{(k + c)^r - (k - 1 + c)^r} \leq 1 - c + d.$$

The proof of this theorem and the next theorems 3.3 and 3.4 are similar to that of theorem 3.1.

Theorem 3.3 *Let* $c \in (0, 1)$, $r > 0$, $0 < a < +\infty$, $d > 0$ *and*

$$L := L_{[\cdot]_c}(a, r, d) := \inf\{E\,[X]_c : 0 \leq X \leq a \text{ a.s.}, \{E\,X^r\}^{1/r} = d\}.$$

Set $n = [a]$.

(I) *If* $0 < d \leq c$ *then* $L = 0$.

(II) *If* $c < a \leq 1 + c$ *and* $c \leq d \leq a$ *then*

$$L = (d^r - c^r)/(a^r - c^r).$$

(III) *If* $0 < r \leq 1$, $n + c < a$ *and determine* $k \in \{0, 1, \ldots, n-1\}$ by $k + c \leq d < k + 1 + c$.

Then $L = k + \dfrac{d^r - (k+c)^r}{(k+1+c)^r - (k+c)^r}$.

(IV) *If* $0 < r \leq 1$, $n + c < a$ *and* $n + c \leq d \leq a$, *then*

$$L = n + \frac{d^r - (n+c)^r}{a^r - (n+c)^r}.$$

(V) If $0 < r \leq 1$, $n + c \geq a$ and determine $k \in \{0, 1, .., n-2\}$ by $k + c \leq d < k + 1 + c$, then

$$L = k + \left[\frac{d^r - (k+c)^r}{(k+1+c)^r - (k+c)^r} \right]$$

(VI) If $0 < r \leq 1$, $n + c \geq a$ and $n - 1 + c \leq d \leq a$, then

$$L = (n-1) + \left[\frac{d^r - (n-1+c)^r}{a^r - (n-1+c)^r} \right].$$

From now on assume $r \geq 1$

Case (A) $n + c < a$

(i) Assume that

$$\frac{1}{a^r - (n+c)^r} \geq \frac{n}{(n+c)^r - c^r}.$$

(i_1) If $c \leq d \leq n + c$, then

$$L = \frac{n \cdot (d^r - c^r)}{((n + c)^r - c^r)}.$$

(i_2) If $n + c \leq d \leq a$, then

$$L = n + \left[\frac{d^r - (n + c)^r}{a^r - (n + c)^r} \right].$$

(ii) Assume that

$$\frac{1}{a^r - (n + c)^r} \leq \frac{n}{(n + c)^r - c^r} \quad \text{and} \quad n \geq 1.$$

If $c \leq d \leq a$, then

$$L = (n + 1) \cdot \left[\frac{d^r - c^r}{a^r - c^r} \right].$$

Case (B) $n + c \geq a$, still $r \geq 1$,

(i) Assume that

$$\frac{1}{a^r - (n - 1 + c)^r} \geq \frac{n - 1}{(n - 1 + c)^r - c^r} \quad \text{and} \quad n \geq 1.$$

(i_1) If $c \leq d \leq n - 1 + c$,

then

$$L = \frac{(n - 1) \cdot (d^r - c^r)}{((n - 1 + c)^r - c^r)}.$$

(i_2) If $n - 1 + c \leq d \leq a$, then

$$L = (n - 1) + \frac{(d^r - (n - 1 + c)^r)}{(a^r - (n - 1 + c)^r)}.$$

(ii) Assume that

$$\frac{1}{a^r - (n-1+c)^r} \leq \frac{n-1}{(n-1+c)^r - c^r} \quad \text{and} \quad n \geq 2 .$$

If $c \leq d \leq a$, then

$$L = n \cdot \frac{(d^r - c^r)}{a^r - c^r} .$$

The last is true also for $a \in \mathbb{N} - \{1\}$.

The case $a = +\infty$ is treated as follows.

Theorem 3.4. *Let* $c \in (0,1)$, $r > 0$, $d > 0$ *and*

$$L := L_{[\cdot]_c}(r, d) := \inf \left\{ E\,[X]_c : X \geq 0 \text{ a.s.}, \{E\,X^r\}^{1/r} = d \right\} .$$

(I) *If* $r > 1$ *then* $L = 0$.

(II) *If* $0 < r \leq 1$, $0 < d \leq c$ *then* $L = 0$.

(III) *If* $r = 1$, $c \leq d < +\infty$ *then* $L = d - c$.

(IV) *If* $0 < r \leq 1$ define $k \in \{0\} \cup \mathbb{N}$ *by* $k + c \leq d < k + 1 + c$. *Then*

$$L = k + \frac{d^r - (k+c)^r}{(k+1+c)^r - (k+c)^r} .$$

Next we pass to Problem 4, see (1.6) – (1.9) .

Theorem 3.5. *Let* $c \in (0,1)$, $0 < r \neq 1$, $0 < a < +\infty$, $d_1 > 0$ $d_r > 0$ *and*

$$U := U_{[\cdot]_c}(a, r, d_1, d_r)$$

$$:= \sup \left\{ E\,[X]_c : 0 \leq X \leq a \text{ a.s.}, E\,X = d_1, \ E\,X^r = d_r \right\}.$$

(I) *Suppose there exists* $\lambda \in [0, 1]$ *such that* $d_1 = \lambda c$, $d_r = \lambda c^r$. *Then* $U = d_1/c$.

(II) *Suppose that* $0 < a \leq c$. *Then* $U = 0$.

(III) *Suppose* $c < a \leq 1 + c$.

(i) *If there exist* $\lambda_1, \lambda_2 \geq 0$ *such that* $\lambda_1 + \lambda_2 \leq 1$, *and* $d_1 = \lambda_1 c + \lambda_2 a$, $d_r = \lambda_1 c^r + \lambda_2 a^r$, *then*

$$U = \frac{(c-a)d_r - (c^r - a^r)\, d_1}{ca^r - c^r a}.$$

(ii) *Suppose* $c \leq d_1 \leq a$ *and assume that either* $r > 1$ and

$$d_1^r \leq d_r \leq \Delta_r := a^r + \left[\frac{a^r - c^r}{a - c} \right](d_1 - a)$$

or $0 < r < 1$ and

$$\Delta_r \leq d_r \leq d_1^r ,$$

then $U = 1$.

From now on suppose $a > 1 + c$ *and set* $n = [a]$,

$$m = \begin{cases} n+1 \text{ ,if } n + c < a \,, \\ n \text{ , if } n + c \geq a \,. \end{cases}$$

(IV) *Suppose* $m - 1 + c \leq d_1 \leq a$ *and assume that either* $r > 1$ *and*

$$d_1^r \leq d_r \leq d_{r,m,a} := a^r + \left[\frac{a^r - (m - 1 + c)^r}{a - (m - 1 + c)} \right] (d_1 - a) \,,$$

or $r \in (0,1)$ *and* $d_{r,m,a} \leq d_r \leq d_1^r$. *Then* $U = m$.

(V) *Suppose one of the following holds:*

(i) $r > 1$ *and there exists* $k \in \{0, 1, \ldots, m-2\}$ *such that* $k + c \leq d_1 < k + 1 + c$ *and* $\tilde{d}_k := (k + c)^r + \left[(k + 1 + c)^r - (k + c)^r \right] \cdot (d_1 - k - c) \leq d_r \leq d_{r,m,c} := c^r + \left[\frac{(m - 1 + c)^r - c^r}{m - 1} \right] \cdot (d_1 - c)$;

(ii) $0 < r < 1$ *and there exists* $k \in \{0, 1, \ldots, m-2\}$ *such that* $k + c \leq d_1 < k + 1 + c$ *and* $d_{r,m,c} \leq d_r \leq \tilde{d}_k$. *Then* $U = d_1 + 1 - c$.

Call

$$\Delta := \frac{m(c^r a - c a^r) - (m - 1)(c^r(m - 1 + c) - c(m - 1 + c)^r)}{(a - c)(m - 1 + c)^r - (m - 1 + c)(a^r - c^r)} \,.$$

(VI) *Assume* $\Delta \geq 1$.

(i) *Suppose there exists* $\lambda_1, \lambda_2 \geq 0$ *with* $\lambda_1 + \lambda_2 \leq 1$ *such that* $d_1 = \lambda_1 c + \lambda_2 (m - 1 + c)$, $d_r = \lambda_1 c^r + \lambda_2 (m - 1 + c)^r$. *Then*

$$U = \frac{d_r(m - 1 + c - m c) - d_1((m - 1 + c)^r - m c^r)}{(m - 1 + c)c^r - (m - 1 + c)^r c}$$

(ii) *Suppose there exist* $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 \leq 1$ *and such that* $d_1 = \lambda_1 (m - 1 + c) + \lambda_2 a$ *and* $d_r = \lambda_1 (m - 1 + c)^r + \lambda_2 a^r$. *Then*

$$U = \frac{m(a - (m - 1 + c))\ d_r - m\ (a^r - (m - 1 + c)^r)\ d_1}{a(m - 1 + c)^r - a^r(m - 1 + c)}.$$

(VII) *Assume* $\Delta \leq 1$.

(i) *Suppose there exists* $\lambda_1, \lambda_2 \geq 0$ *with* $\lambda_1 + \lambda_2 \leq 1$ *such that* $d_1 = \lambda_1 c + \lambda_2 a$, $d_r = \lambda_1 c^r + \lambda_2 a^r$. *Then*

$$U = \frac{d_1(m c^r - a^r) + d_r(a - cm)}{ac^r - c a^r}.$$

(ii) *Suppose there exists* $\lambda_1, \lambda_2, \lambda_3 \geq 0$ *with* $\lambda_1 + \lambda_2 + \lambda_3 = 1$ *and such that*

$d_1 = \lambda_1 c + \lambda_2 (m - 1 + c) + \lambda_3 a$,

$d_r = \lambda_1 c^r + \lambda_2 (m - 1 + c)^r + \lambda_3 a^r$.

Then

$$U = 1 + \frac{(m - 1)(a - c - m + 1)(d_r - c^r) - (m - 1)(a^r - (m - 1 + c)^r)(d_1 - c)}{(a - c)((m - 1 + c)^r - c^r) - (m - 1)(a^r - c^r)}.$$

Proof. Observe that the points

$U_k = (k + c, (k + c)^r, k + 1)$, $k \in N \cup \{0\}$ belong to the plane

$Q: z = x + 1 - c$. Set $n := [a]$. We meet the cases

1) $\underline{n + c < a}$

Here the points of the graph $(t, t^r, [t]_c)$, $0 \le t \le a$, $0 < r \neq 1$ that determine the upper envelope of its convex hull are: $0 = (0,0,0)$, U_k; $k = 0,1,...n$ and $B = (a, a^r, n + 1)$.

2) $\underline{n + c \ge a}$

Same as in #1 , however $k = 0,1,...,n-1$ and $B = (a, a^r, n)$. Therefore the last point of the graph will be $B = (a, a^r, m)$, where

$$m := \begin{cases} n+1, & \text{if } n+c < a \\ n, & \text{if } n+c \ge a \end{cases}.$$

Hence $k = 0,1,...,m-1$ in either case of $n + c < a$ or $n + c \ge a$. Thus in either case, the upper envelope of convex hull of $(t, t^r, [t]_c)$, $0 \le t \le a$, $0 < r \neq 1$ is determined by the points O, U_k; $k = 0,1,...,m-1$ and B . Set $A := U_{m-1} = (m - 1 + c, (m - 1 + c)^r, m)$, $D := U_0 = (c, c^r, 1)$

Next we observe that the part S we are interested in, of the upper-envelope of the associated convex hull is a concave surface formed by the following planes:

$\underline{\text{Case } \Delta \le 1}$. $Q : z = x + 1 - c$, $P : z = m$, $\overset{\Delta}{(DAB)}$, $\overset{\Delta}{(ODB)}$.

$\underline{\text{Case } \Delta > 1}$. Q, P, $\overset{\Delta}{(ODA)}$, $\overset{\Delta}{(OAB)}$

The values of U in the different arising cases are now clearly given by the geometric moment theory.

The case $a = +\infty$ is similar and we shall state the result without proof.

Theorem 3.6. *Let* $0 < c < 1$, $0 < r \neq 1$, $d_1 > 0$, $d_r > 0$ *and*

$$U = U_{[\cdot]_c}(r, d_1, d_r) := \sup\left\{E[X]_c : X \geq 0 \text{ a.s., } E X = d_1, E X^r = d_r\right\}.$$

(I) *Suppose* $r > 1$. (i) *Let* $0 < d_1 \leq c$ and $c^r \leq d_r < \infty$ *then* $U = d_1/c$.

(ii) *Suppose that there exist* $\lambda_1, \lambda_2 \geq 0$ *with* $\lambda_1 + \lambda_2 \leq 1$ *and such that* $d_1 = \lambda_1 c$, $d_r = (\lambda_1 + \lambda_2)c^r$. *Then* $U = d_1/c$.

(iii) *Suppose there exists integer* k *such that* $k + c \leq d_1 < k + 1 + c$ *and* $(k + c)^r + \left[(k + 1 + c)^r - (k + c)^r\right](d_1 - k - c) \leq d_r < \infty$.

Then $U = d_1 + 1 - c$.

(II) *Suppose* $0 < r < 1$.,

(i) *Let* $c \leq d_1 < \infty$, $0 < d_r \leq c^r$. *Then* $U = d_r c^{-r}$.

(ii) *Suppose there exist* $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, *such that* $\lambda_1 + \lambda_2 \leq 1$ *and* $d_1 = (\lambda_1 + \lambda_2)c$, $d_r = \lambda_2 c^r$. *Then* $U = d_r c^{-r}$.

(iii) *Suppose that there exists an integer* k *with* $k + c \leq d_1 < k + 1 + c$ *and such that* $c^r < d_r \leq (k + c)^r + \left[(k + 1 + c)^r - (k + c)^r\right](d_1 - k - c)$. *Then* $U = d_1 + 1 - c$.

The next two theorems deal with the lower bound in Problem 4, see (1.7), (1.8), (1.9).

Theorem 3.7. *Let* $a > 0$, $0 < c \le 1$, $d_1 > 0$, $d_r > 0$, $0 < r \ne 1$ *Denote*

$$L := L_{[\cdot]_c}(a, r, d_1, d_r)$$
$$:= \inf\left\{E[X]_c : 0 \le X \le a \text{ a.s. and } E X = d_1, E X^r = d_r\right\}.$$

(I) *Suppose there exist* t_1, t_2, λ *such that* $0 \le t_1 \le t_2 \le \min(c, a)$, $0 \le \lambda \le 1$ *and* $d_1 = (1-\lambda)\, t_1 + \lambda t_2$, $d_r = (1-\lambda) t_1^r + \lambda\, t_2^r$.

Then $L = 0$.

(II) *Suppose* $c < a \le 1 + c$ *and there exist* $\lambda_1, \lambda_2 \ge 0$ *such that* $\lambda_1 + \lambda_2 \le 1$ *and* $d_1 = \lambda_1 c + \lambda_2 a$, $d_r = \lambda_1 c^r + \lambda_2 a^r$.

Then

$$L = (d_r - c^{r-1} d_1) / (a^r - a\, c^{r-1}).$$

From now on assume $a > 1 + c$.

Set $n = [a]$ *and*

$$m = \begin{cases} n+1, & \text{if } n + c < a \\ n, & \text{if } n + c \ge a \end{cases}.$$

(III) *Suppose that one of the following two condtions holds:*

(i) $r > 1$ *and there exists* $k = \{0, 1, \ldots, m-2\}$ *such that* $k + c \le d_1 < k + 1 + c$ *and* $\tilde{d}_k := (k+c)^r + ((k+1+c)^r - (k+c)^r)(d_1 - k - c) \le d_r \le d_{r,m,c} := c^r + \left[\frac{(m-1+c)^r - c^r}{m-1} \right] \cdot (d_1 - c)$;

(ii) $0 < r < 1$ *and there exists* $k \in \{0, 1, \ldots, m-2\}$ *such that* $k + c \le d_1 < k + 1 + c$ *and* $d_{r,m,c} \le d_r \le \tilde{d}_k$. *Then* $L = d_1 - c$.

(IV) *Assume that*

$$\Delta_{r,m,c} := \frac{(m-1)(a^r - c^{r-1}\, a)}{(m-1+c)^r - c^{r-1}(m-1+c)} \le m .$$

(i) *Suppose there exist* $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 \le 1$ *such that* $d_1 = \lambda_1 c + \lambda_2(m-1+c)$ *and* $d_r = \lambda_1 c^r + \lambda_2(m-1+c)^r$. *Then*

$$L = \frac{(m-1)(d_r - c^{r-1}\, d_1)}{(m-1+c)^r - c^{r-1}(m-1+c)} .$$

(ii) *Suppose there exist* $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 \le 1$ *such that* $d_1 = \lambda_1(m-1+c) + \lambda_2 a$, $d_r = \lambda_1(m-1+c)^r + \lambda_2 a^r$. *Then*

$$L = \frac{((m-1+c)m - a(m-1))d_r - ((m-1+c)^r m - a^r(m-1))d_1}{(m-1+c)a^r - a(m-1+c)^r} .$$

(V) *Assume that* $\Delta_{r,m,c} > m$.

(i) *Suppose that there exists* $\lambda_1, \lambda_2 \geq 0$, with $\lambda_1 + \lambda_2 \leq 1$ *and such that* $d_1 = \lambda_1 c + \lambda_2 a$, $d_r = \lambda_1 c^r + \lambda_2 a^r$. *Then*

$$L = \frac{m(d_r - c^{r-1} d_1)}{a^r - ac^{r-1}}.$$

(ii) *Suppose that there exists* $\lambda_1, \lambda_2, \lambda_3 \geq 0$ *with* $\lambda_1 + \lambda_2 + \lambda_3 = 1$ *and such that* $d_1 = \lambda_1 c + \lambda_2 (m - 1 + c)^r + \lambda_3 a$,

$$d_r = \lambda_1 c^r + \lambda_2 (m - 1 + c)^r + \lambda_3 a^r.$$

Then

$$L = \frac{(m-1)(m-a+c)(d_r - c^r) - (m(m-1+c)^r - (m-1)a^r - c^r)(d_1 - c)}{(m-1)(a^r - c^r) - (a-c)((m-1+c)^r - c^r)}.$$

Proof: Observe that the points

$w_k = (k + c, (k + c)^r, k)$, $k \in N \cup \{0\}$ belong to the same plane $Q : z = x - c$

Set $n = [a]$. We meet the cases

(1) $n + c < a$. Here the points of the graph $(t, t^r, [t]_c)$, $0 \leq t \leq a$, $0 < r \neq 1$ that determine the lower envelope of its convex hull are:

$O = (0, 0, 0)$, w_k; $k = 0, 1, \ldots, n$ and

$B = (a, a^r, n + 1)$.

(2) $n + c \geq a$. Same as in (1), however $k = 0, 1,.., n - 1$ and $B = (a, a^r, n)$. Therefore, the last point of the graph will be $B = (a, a^r, m)$, where

$$m := \begin{cases} n + 1 , & \text{if } n + c < a \\ n , & \text{if } n + c \geq a \end{cases}$$

Hence, $k = 0, 1,..., m - 1$ in either case of $n + c < a$ or $n + c \geq a$.

Thus, in either case, the lower envelope of convex hull of $(t, t^r, [t]_c)$, $0 \leq t \leq a$, $0 < r \neq 1$ is determined by the points O , w_k; $k = 0, 1,..., m - 1$ and B . Set $A := w_{m-1} = (m - 1 + c, (m - 1 + c)^r, m - 1)$ and $W := w_0 = (c, c^r, 0)$.

Next we observe that the part S we are interested in, of the lower envelope of the associated convex hull is a convex surface focused by the following convex hulls:

Case $\Delta_{r,m,c} \leq m$: conv. hull $(\widehat{OW})$, conv. hull $(\widehat{WA})$, conv. hull $(\overset{\Delta}{OWA})$ and conv. hull $(\overset{\Delta}{OAB})$.

Case $\Delta_{r,m,c} \geq m$: conv. hull $(\widehat{OW})$, conv.hull $(\widehat{WA})$, conv.hull $(\overset{\Delta}{WOB})$ and conv.hull $(\overset{\Delta}{WAB})$.

The values of L in the different arising cases are now clearly given by the geometric moment method. □

Theorem 3.7 for $a = +\infty$ reads as follows. The proof is similar to that of Theorem 3.7 and thus omitted.

Theorem 3.8. *Let* $0 < c < 1$, $d_1 > 0$, $d_r > 0$, $0 < r \neq 1$.

Denote

$$L := L_{[\cdot]_c}(r, d_1, d_r) := \inf\{E\,[X]_c : X \geq 0 \text{ a.s.}, E\,X = d_1, E\,X^r = d_r\}$$

(I) *Suppose one of the following three conditions holds:*

(i) *there exist* t_1, t_2, λ *such that* $0 \leq t_1 \leq t_2 \leq c$, $0 \leq \lambda \leq 1$, *such that*

$$d_1 = (1-\lambda)t_1 + \lambda t_2, \; d_r = (1-\lambda)t_1^r + \lambda t_2^r;$$

(ii) $r > 1$ *and there exist* $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 \leq 1$ *such that* $d_1 = \lambda_1 c$, $d_r = (\lambda_1 + \lambda_2)c^r$;

(iii) $r > 1$ *and* $0 < d_1 \leq c$ *and* $d_r \geq c^r$.

Then $L = 0$.

(II) *Suppose there exists integer* k *such that* $k + c \leq d_1 < k + 1 + c$ *and either when* $r > 1$ *we have* $d_r \geq d_{r,k} := (k+c)^r + ((k+1+c)^r - (k+c)^r)(d_1 - k - c)$; *or when* $0 < r < 1$ *we have* $d_{r,k} \geq d_r \geq c^r$. *Then* $L = d_1 - c$

(III) *Let* $0 < r < 1$, *and either there exist* $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_1 + \lambda_2 \leq 1$ *such that* $d_1 = (\lambda_1 + \lambda_2)c$, $d_r = \lambda_2 c^r$, or $d_1 \geq c$ *and* $0 < d_r \leq c^r$. *Then*

$$L = (c^{r-1}d_1 - d_r)c^{1-r} .$$ ∎

In case of conventional (Jefferson) rounding or MYZ–rounding, $[x] := [x]_1$, *the last four theorems have a simpler form. We shall only state the results.*

Theorem 3.9. *Let* $a > 0$, $0 < r \neq 1$, $d_1 > 0$, $d_r > 0$ *and*

$$U = U_{[\cdot]}(a, r, d_1, d_r) := \sup\{E[X] : 0 \le X \le a \text{ a.s. and } EX = d_1, EX^r = d_r\} .$$

Call $\theta = [a]$.

(I) *Denote*

$$\Delta_{r,a} := a^r + \frac{a^r - \theta^r}{a - \theta}(d_1 - \theta) .$$

Suppose that $a \neq \theta$, $\theta \le d_1 \le a$ *and either* $r > 1$, $d_1^r \le d_r \le \Delta_{r,a}$, *or* $0 < r < 1$ and $\Delta_{r,a} \le d_r \le d_1^r$. *Then* $U = \theta$.

(II) *Suppose* $0 < \theta \neq a$ *and there are* $\lambda_1, \lambda_2 \ge 0$ *with* $\lambda_1 + \lambda_2 \le 1$ *and such that* $d_1 = \lambda_1\theta + \lambda_2 a$ *and* $d_r = \lambda_1\theta^r + \lambda_2 a^r$. *Then*

$$U = \frac{(\theta^r - a^r)d_1 + (a - \theta)d_r}{a(\theta^{r-1} - a^{r-1})} .$$

(III) *Let* $\theta \ge 1$ *and there exists* $k \in \{0, 1, \ldots, \theta - 1\}$ *such that* $k \le d_1 < k + 1$ *and either* $r > 1$ *and* $d_{r,k} := k^r + \left[(k+1)^r - k^r\right](d_1 - k) \le d_r \le \theta^{r-1} d_1$ *or* $0 < r < 1$ *and* $\theta^{r-1} d_1 \le d_r \le d_{r,k}$. *Then* $U = d_1$. ∎

For the case $a = \infty$ we have the following version of the above theorem.

Theorem 3.10. *Let* $0 < r \neq 1$, $d_1 > 0$, $d_r > 0$ *and* $U := U_{[\cdot]}(r, d_1, d_r)$ $:= \sup\{E[X] : X \geq 0 \text{ a.s. and } E X = d_1, E X^r = d_r\}$.

Suppose there exists a nonegative integer k *such that* $k \leq d_1 < k + 1$ *and either* $r > 1$ *and* $d_r \geq d_{r,k} := k^r + \left[(k+1)^r - k^r\right](d_1 - k)$ *or* $0 < r < 1$ *and* $0 < d_r \leq d_{r,k}$. *Then* $U = d_1$. ∎

If we change in theorems 3.9 and 3.10 the upper bound U with the corresponding lower bound (see (1.7), (1.8), (1.9) for $c = 1$) we obtain the following two theorems.

Theorem 3.11 *Let* $a > 0$, $0 < r \neq 1$, $d_1 > 0$, $d_r > 0$ *and*

$$L := L_{[\cdot]}(a, r, d_1, d_r)$$

$$:= \inf\{E[X] : 0 \leq X \leq a \text{ a.s.}, E X = d_1, E X^r = d_r\}$$

(I) *Suppose there exist* t_1, t_2, λ *such that* $0 \leq t_1 \leq t_2 \leq 1$, $0 \leq \lambda \leq 1$, *and* $d_1 = (1 - \lambda) t_1 + \lambda t_2$ *and* $d_r = (1 - \lambda) t_1^r + \lambda t_2^r$. *Then* $L = 0$.

(II) *If* $0 < a \leq 1$ *then* $L = 0$.

(III) *If* $1 < a < 2$ *and there exist* $\lambda_1, \lambda_2 \geq 0$ *with* $\lambda_1 + \lambda_2 \leq 1$ *and such that* $d_1 = \lambda_1 + \lambda_2 \alpha$, $d_r = \lambda_1 + \lambda_2 a^r$. *Then* $L = (d_r - d_1) / (\alpha^r - a)$.

From now on assume $a \geq 2$, *and let* $\theta = [a]$.

(IV) *Suppose* $\Delta_\theta := (\theta - 1)(a^r - a) / (\theta^r - \theta) \leq \theta$.

(i) *Suppose* $d_1 = \lambda_1 + \lambda_2\theta$, $d_r = \lambda_1 + \lambda_2\theta^r$ *for some* $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 \leq 1$. *Then* $L = \Delta_\theta$.

(ii) *Suppose* $d_1 = \lambda_1\theta + \lambda_2 a$ *and* $d_r = \lambda_1\theta^r + \lambda_2 a^r$ *for some* $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 \leq 1$. *Then*

$$L = \frac{(\theta^2 - a(\theta - 1))d_r - (\theta^{r+1} - \alpha^r(\theta - 1))d_1}{\theta a^r - a\theta^r}.$$

(V) *Suppose* $\Delta_\theta > \theta$.

(i) *Suppose* $d_1 = \lambda_1 + \lambda_2\theta + \lambda_3 a$ *and* $d_r = \lambda_1 + \lambda_2\theta^r + \lambda_3 a^r$ *for some* $\lambda_1, \lambda_2, \lambda_3 \geq 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$. *Then*

$$L = \frac{(\theta - 1)(\theta - a + 1)(d_r - 1) - ((\theta^r - 1)\theta - (a^r - 1)(\theta - 1))(d_1 - 1)}{(\theta - 1)(a^r - 1) - (a - 1)(\theta^r - 1)}$$

(ii) *Suppose* $d_1 = \lambda_1 + \lambda_2 a$, $d_r = \lambda_1 + \lambda_2 a^r$ *for some* $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 \leq 1$. *Then* $L = \theta(d_r - d_1)/(a^r - a)$.

(VI) *Suppose* $\theta > 1$ *and one of the following holds.*

(i) $r > 1$ *and there exists* $k \in \{1, \ldots, \theta - 1\}$ *such that* $k \leq d_1 < k + 1$ *and*

$$d_{r,k} := k^r + ((k + 1)^r - k^r)(d_1 - k) \leq d_r \leq$$

$$\Delta_{r,\theta} := 1 + (\theta^r - 1)(d_1 - 1)/(\theta - 1) ;$$

(ii) $0 < r < 1$ *and there exists* $k \in \{1, ..., \theta - 1\}$ *such that* $k \le d_1 < k + 1$ *and* $\Delta_{r,\theta} \le d_r \le d_{r,k}$. *Then* $L = d_1 - 1$. ∎

The special case $a = \infty$ is treated as follows.

Theorem 3.12 *Let* $0 < r \ne 1$, $d_1 > 0$, $d_r > 0$ *and* $L := L_{[\cdot]}(r, d_1, d_r) := \inf\{E[X] : X \ge 0 \text{ a.s.}, E X = d_1, E X^r = d_r\}$.

(I) *Suppose there exist* $0 \le t_1 \le t_2 \le 1$, $0 \le \lambda \le 1$ *such that* $d_1 = (1 - \lambda)t_1 + \lambda t_2$ and $d_r = (1 - \lambda)t_1^r + \lambda t_2^r$. *Then* $L = 0$.

(II) *Suppose* $0 < r < 1$ *and either* $d_1 = \lambda_1 + \lambda_2$, $d_r = \lambda_1$, $\lambda_i \ge 0$, $i = 1, 2$, $\lambda_1 + \lambda_2 \le 1$ *or* $d_1 \ge 1$, $0 < d_r \le 1$. *Then* $L = d_1 - d_r$.

(III) *Suppose* $0 < r < 1$ *and for some positive integer* k, $k \le d_1 < k + 1$, *and* $1 \le d_r \le (k^r + ((k + 1)^r - k^r))(d_1 - k)$. *Then* $L = d_1 - 1$.

(IV) *Suppose* $r > 1$ *and either* $d_1 = \lambda_1$, $d_r = \lambda_1 + \lambda_2$, $\lambda_i \ge 0$, $i = 1, 2$, $\lambda_1 + \lambda_2 \le 1$, *or* $0 < d_1 \le 1$ *and* $1 \le d_r$. *Then* $L = 0$.

(V) *Suppose* $r > 1$ *and for some positive integer* k, $k \le d_1 < k + 1$, *and* $(k^r + ((k + 1)^r - k^r))(d_1 - k) \le d_r$. *Then* $L = d_1 - 1$. ∎

Up to now we always assume that the divisor point $c \in (0, 1)$. The rest of this section is devoted to the Adams rule of rounding $\lceil \cdot \rceil = [\cdot]_0$. We state the results again without proofs.

Theorem 3.13 *Let* $a > 0$, $0 < r \neq 1$, $d_1 > 0$, $d_r > 0$.

Denote

$$U := U_{\lceil \cdot \rceil}(a, r, d_1, d_r)$$

$$= \sup\{E\lceil X\rceil : 0 \le X \le \alpha \text{ a.s.}, E X = d_1, E X^r = d_r\}.$$

Set $\theta = \lceil a \rceil$.

(I) *Suppose that* $\theta - 1 \le d_1 \le a$ *and either*

(I1) $r > 1$ *and* $d_1^r \le d_r \le \Delta_r := a^r + \left[\frac{a^r - (\theta - 1)^r}{a - \theta + 1}\right](d_1 - \theta)$,

or

(I2) $0 < r < 1$ *and* $\Delta_r \le d_r \le d_1^r$. *Then*

$U = \theta$.

(II) *Suppose* $\theta > 1$ *and there are* $\lambda_1, \lambda_2 \ge 0$ *with* $\lambda_1 + \lambda_2 = 1$ *such that* $d_1 = \lambda_1(\theta - 1) + \lambda_2 a$, $d_r = \lambda_1(\theta - 1)^r + \lambda_2 a^r$. *Then*

$$U = 1 + \frac{((\theta - 1)^r - a^r)\, d_1 + (a - \theta + 1)\, d_r}{a(\theta - 1)^{r-1} - a^r}.$$

(III) *Suppose* $\theta > 1$, *and let* $k := [d_1]$ *and*

$$\Delta_r := k^r + ((k + 1)^r - k^r)(d_1 - k), \quad \tilde{\Delta}_r := (\theta - 1)^{r-1} d_1.$$

Suppose that either

(III 1) $r > 1$ *and* $\Delta_r \le d_r \le \tilde{\Delta}_r$

or

(III 2) $0 < r < 1$ *and* $\tilde{\Delta}_r \le d_r \le \Delta_r$.

Then

$$U = d_1 + 1 .$$

The case $a = \infty$ is considerably simpler.

Theorem 3.14 *Let* $0 < r \neq 1$, $d_1, d_r > 0$. *Denote* $U := U_{\lceil \cdot \rceil}(r, d_1, d_r)$

$= \sup \{E \lceil X \rceil : X \ge 0 \text{ a.s.}, E X = d_1, E X^r = d_r\}$.

Denote $k := [d_1]$ *and* $\Delta_r := k^r + \left[(k+1)^r - k^r\right](d_1 - k)$.

Suppose that either

(I) $r > 1$, $d_r \ge \Delta_r$,

or

(II) $0 < r < 1$, $d_r \le \Delta_r$.

Then $U = d_1 + 1$.

Theorem 3.15 *Let* $0 < r \neq 1$, $d_1, d_r > 0$, $a > 0$. *Denote*

$$L := L_{\lceil \cdot \rceil}(a, r, d_1, d_r)$$

$= \inf \{E \lceil X \rceil : 0 \le X \le a \text{ a.s. and } E X = d_1, E X^r = d_r\}$.

Call $\theta = \lceil a \rceil$.

(I) *Suppose* $0 < a \le 1$ *and* $d_1 = \lambda a$, $d_r = \lambda a^r$ *for some* $\lambda \in [0,1]$. *Then* $L = d_1 / a$.

(II) *Let* $a > 1$ *and suppose* $d_1 = \lambda_1 (\theta - 1) + \lambda_2 a$, $d_r = \lambda_1 (\theta - a)^r + \lambda_2 a^r$ *for some* $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 \leq 1$. *Then* $\mathbf{L} = \dfrac{(a^r - \theta (\theta - 1)^{r-1}) d_1 + (\theta - a) d_r}{a^r - a (\theta - a)^{r-1}}$.

(III) *Suppose one of the following holds:*

(i) $r > 1$ *and for some* $k \in \{0, 1, \ldots, \theta - 2\}$, $k \leq d_1 < k + 1$, $\Delta_r := k^r + \left[(k+1)^r - k^r\right](d_1 - k) \leq d_r \leq (\theta - 1)^{r-1} d_1$;

(ii) $0 < r < 1$ *and for some* $k \in \{0, 1, \ldots, \theta - 2\}$, $k \leq d_1 < k + 1$, $(\theta - 1) d_1^{r-1} \leq d_r \leq \Delta_r$.

Then $\mathbf{L} = d_1$.

The case $a = \infty$ is treated as follows.

Theorem 3.16 *Let* $0 < r \neq 1$, $d_1 > 0$, $d_r > 0$. *Denote*

$$\mathbf{L} := \mathbf{L}_{\lceil . \rceil}(r, d_1, d_r) := \inf \{\mathbf{E} \lceil X \rceil : X \geq 0 \text{ a.s. and } \mathbf{E} X = d_1, \mathbf{E} X^r = d_r\}.$$

Then case (III) *from Theorem 3.15 takes place.*

Similar to theorems 3.13 – 3.16 we have the following solution of moment problem 3 for the Jefferson rule of rounding.

Theorem 3.17 *Let* $d > 0$, $r > 0$.

(A) *Let* $0 < a < \infty$. *Consider*

$$U = U_{\lceil \cdot \rceil}(a, r, d) = \sup \{E \lceil X \rceil : 0 \le X \le a \text{ a.s. } E X^r = d^r\},$$

and

$$L = L_{\lceil \cdot \rceil}(a, r, d) = \inf \{E \lceil X \rceil : 0 \le X \le a \text{ a.s. } E X^r = d^r\},$$

(A1) *Denote* $n = \lceil a \rceil$, $k = \lceil d \rceil$.

(I) *If* $k = n$ *then* $U = n$.

(II) *If* $r \le 1$ *and* $k < n$ *then*

$$U = 1 + d^r (n-1)^{1-r}.$$

(III) *If* $r \ge 1$ *and* $k < n$ *then*

$$U = k + \theta_k \le 1 + d$$

where

$$\theta_k := (d^r - (k-1)^r)/(k^r - (k-1)^r)$$

(A2) *Let again* $n = \lceil a \rceil$.

(I) *If* $r \ge 1$ *and* $\tilde{a}_r := (n-1)^{1-r} (a^r - (n-1)^r) \ge 1$, *then* $L = n(d/a)^r$.

(II) *If* $r \ge 1$ *and* $\tilde{a}_r \le 1$, *then*

$$L = \begin{cases} d^r (n-1)^{1-r}, & \text{when } 0 < d < n-1 \\ (n-1) + \dfrac{d^r - (n-1)^r}{a^r - (n-1)^r}, & \text{when } n-1 \le d \le a. \end{cases}$$

(III) *If* $0 < r \le 1$ *and if for some* $k \in \{0, 1, \ldots, n-2\}$, $k \le d < k+1$, *then*

$L = k + (d^r - k^r)/((k+1)^r - k^r)$.

(IV) *If* $0 < r \le 1$ *and* $n - 1 \le d \le a$ *then* $L = n + (d^r - a^r)/(d^r - (n-1)^r)$.

(B) *Let* $a = \infty$ *and denote*

$$U := U_{\lceil \cdot \rceil}(r, d) := \sup \{E \lceil X \rceil : X \geq 0 \text{ a.s.}, \ E X^r = d^r\}$$

and

$$L := L_{\lceil \cdot \rceil}(r, d) := \inf\{E \lceil X \rceil : X \geq 0 \text{ a.s.}, \ E X^r = d^r\}$$

(B1) *Denote* $k = \lceil d \rceil$.

(I) *If* $r \geq 1$ *then* $U = k + \theta_k \leq 1 + d$, *where* $\theta_k := (d^r - (k-1)^r)/(k^r - (k-1)^r)$.

(II) *If* $r < 1$ *then* $U = +\infty$.

(B2) (I) *If* $r > 1$ *then* $L = 0$.

(II) *If* $r = 1$ *then* $L = d$.

(III) *If* $0 < r < 1$ *and* $k := \lceil d \rceil$ *then*

$$L = k + (d^r - k^r)/((k+1)^r - k^r) .$$

4. **Closeness of random processes with fixed moment characteristics.**

The moment problems we are going to consider in this section may be viewed as extensions of Problem 1 (see (1.1) – (1.3)) for measures μ generated by the joint distribution of random processes. Namely, let the class $\mathcal{M}$, the space $\mathcal{S}(C[0, 1])$ and the set $T = \{t_1, ..., t_N\}$ be defined as in section 1. The subject of this section is

Moment Problem 5. Given $h, g_{i,j} \in \mathcal{M}$ $(i = 1, ..., N, j = 1, ..., n)$ find all possible values of $E\, h(\|X - Y\|)$ when the moments

$$(4.1) \qquad E g_{ij}(|X(t_i)|) = a_{ij}, \ E\, g_{ij}(|Y(t_i)|) = b_{ij}, \ i = 1, ..., N, \ j = 1, ..., n .$$

are given. In particular, determine the bounds

$$(4.2) \qquad \inf \mathbf{E}\, h(\|X - Y\|)\,,\ \sup \mathbf{E}\, h(\|X - Y\|)\,,$$

subject to the constraints (4.1).

In other words, suppose we observe two continuous processes, X and Y only through the "window" T. Suppose for each point of the "window" we know some moment characteristics of X and Y. The problem is to determine the possible deviations between the processes outside the window. In particular, "what is the minimal distance between X and Y with given moment information (4.1)?" is just a special case of the moment problem 5.

We start with the case $n = 1$ in (4.1), i.e. given $h, g_i \in \mathcal{M}\ (i = 1,\dots, N)$ and assuming that

$$(4.3) \qquad \mathbf{E}\, g_i(|X(t_i)|) = a_i\,,\ \mathbf{E}\, g_i(|Y(t_i)| = b_i\,,\ i = 1,\dots, N\,,$$

we are interested in the range of $\mathbf{E}\, h(\|X - Y\|)$. The solution of this problem will be given under some assumptions of the following type:

$A(h, g)$: $h \circ g^{-1}(t)$ $(t \geq 0)$ is a convex function. (Here and in the sequel, f^{-1} stands for the inverse of $f \in \mathcal{M}$);

$B(g)$: $g^{-1}(\mathbf{E}\, g(|\xi + \eta|)) \leq g^{-1}(\mathbf{E}\, g(|\xi|)) + g^{-1}(\mathbf{E}g(|\eta|))$ for any $\xi, \eta \in \mathcal{S}(\mathbf{R})$ (Here, $\mathcal{S}(\mathbf{R})$ is the set of all real–valued r.v.'s.);

$C(g)$: $\mathbf{E}\, g(|\xi + \eta|) \leq \mathbf{E}\, g(|\xi|) + \mathbf{E}\, g(|\eta|)$ for any $\xi, \eta \in \mathcal{S}(\mathbf{R})$;

$D(h, g)$: $\lim_{t \to \infty} h(t)/g(t) = 0$.

Remark 4.1 If $h(t) = t^p$, $g(t) = t^q$ $(p > 0, q > 0)$ then $A(h, g) \Leftrightarrow p \geq q$, $B(g) \Leftrightarrow q \geq 1$, $C(q) \Leftrightarrow q \leq 1$, $D(h, g) \Leftrightarrow q > p$. □

Let $T = \{0 \leq t_1 \leq \ldots \leq t_N \leq 1\}$, $\bar{a} = (a_1, a_2, \ldots, a_N) \in R_+^N$, $\bar{b} = (b_1, b_2, \ldots, b_N) \in R_+^N$ and $\bar{g} = (g_1, g_2, \ldots, g_N) \in \mathcal{M}$ be fixed vectors. Denote by $\mathcal{S}(T, \bar{g}, \bar{a})$, the space of all $X \in \mathcal{S}(C[0, 1])$ satisfying the marginal moment conditions $E\, g_i(|X(t_i)|) = a_i$ $(i = 1, \ldots, N)$, and let

$$I\{h, \bar{g}, T, \bar{a}, \bar{b}\} := \inf\{E\, h(\|X - Y\|) : X \in \mathcal{S}(T, \bar{g}, \bar{a}) , Y \in \mathcal{S}(T, \bar{g}, \bar{b})\} . \tag{4.4}$$

In the next four theorems we describe the exact range of values of $E\, h(\|X - Y\|)$ under different conditions of type A–D.

Theorem 4.1 *Let* $A(h, g_i)$ *and* $B(g_i)$ *hold for any* $i = 1, 2, \ldots, N$. *Then*

(i)

$$I\{h, \bar{g}, T, \bar{a}, \bar{b}\} = \sup_{1 \leq i \leq N} h(|g_i^{-1}(a_i) - g_i^{-1}(b_i)|) \tag{4.5}$$

and,

(ii) *for any* $\nu \geq I\{h, g, T, \bar{a}, \bar{b}\}$ *there exist random processes* $X_\nu \in \mathcal{S}(T, \bar{g}, \bar{a})$ *and* $Y_\nu \in \mathcal{S}(T, \bar{g}, \bar{b})$ *such that*

$$E\, h(\|X_\nu - Y_\nu\|) = \nu . \tag{4.5}$$

Proof Claim 1 $I\{h, \bar{g}, T, \bar{a}, \bar{b}\} \geq \sup_{1 \leq i \leq N} \phi_i(a_i, b_i)$, *where* $\phi_i(a_i, b_i) := h(|g_i^{-1}(a_i) - g_i^{-1}(b_i)|)$.

Let $X, Y \in \mathcal{S}(C[0, 1])$ and $\xi := g(|X(t_i) - Y(t_i)|)$. Then, by the Jensen's inequality and $A(h, g_i)$

$$h^{-1}(E\, h(\|X - Y\|)) \geq h^{-1}(E\, h(|X(t_i) - Y(t_i)|))$$

$$= h^{-1}(E\, h \circ g_i^{-1}(\xi)) \geq h^{-1} \circ h \circ g_i^{-1}\, E(\xi) = g_i^{-1}\, E(\xi).$$

Further, by $B(g_i)$,

$$h \circ g_i^{-1}(E(\xi))$$

$$\geq h\left[\left(|g_i^{-1}(E\, g_i(|X(t_i)|)) - g_i^{-1}(E\, g_i(|Y(t_i)|))|\right)\right]$$

$$= h(|g_i^{-1}(a_i) - g_i^{-1}(b_i)|)$$

which proves the claim.

Claim 2. *The infimum in the left-hand side of* (4.4) *is attained and* (4.5) *holds.*

Define $\tilde{X}, \tilde{Y} \in \mathcal{S}(C[0, 1])$ to be random polygon lines with vertices at the points $0, t_1, \ldots, t_N, 1$ given by

$$(4.6) \qquad \begin{cases} \tilde{X}(t_i, \omega) = g^{-1}(a_i), \quad \tilde{Y}(t_i, \omega) = g^{-1}(b_i)., i = 1,.., N, \\ \tilde{X}(0, \omega) = \tilde{Y}(0, \omega) = 0, \text{ if } t_1 > 0, \\ \tilde{X}(1, \omega) = \tilde{Y}(1, \omega) = 0, \text{ if } t_N < 1, \ \omega \in \Omega \end{cases}$$

For any $\omega \in \Omega$

$$\|\tilde{X}(\cdot, \omega) - \tilde{Y}(\cdot, \omega)\| = \sup_{1 \le i \le N} |g_i^{-1}(a_i) - g_i^{-1}(b_i)|$$

and hence

$$\text{(4.7)} \qquad E\, h(\|\tilde{X} - \tilde{Y}\|) = \sup_{1 \le i \le N} \phi_i(a_i, b_i) .$$

Further, by (4.6),

$$\text{(4.8)} \qquad \tilde{X} \in \mathcal{S}(T, \bar{g}, \bar{a}) , \tilde{Y} \in \mathcal{S}(T, \bar{g}, \bar{b}) .$$

Invoking (4.7), (4.8) and Claim 1 we complete the proof of the claim.

Claim 3 (ii) *is satisfied.*

Let $\tau \in (0, 1)$, $t \notin T$. Define the r.v.'s X_ν and Y_ν in $\mathcal{S}(C[0, 1])$ as follows:

$$\text{(4.9)} \qquad X_\nu(t) = \tilde{X}(t) , Y_\nu(t) = \tilde{Y}(t) \text{ for } t = 0, t_1, \ldots, t_N, 1$$

(see (4.6)),

$$\text{(4.10)} \qquad X_\nu(\tau) = h^{-1}(\nu) , Y_\nu(\tau) = 0$$

and let $X_\nu(t)$, $Y_\nu(t)$ be random polygon line with vertices at $0, t_1, \ldots, t_N, 1$ and τ. Invoking now Claim 2 we have that

$$\|X_\nu - Y_\nu\| = h^{-1}(\nu) \ge \sup_{1 \le i \le N} |g_i^{-1}(a_i) - g_i^{-1}(b_i)|$$

and thus (4.5) holds.

Theorem 4.2 *Let* $A(h, g_i)$ *and* $C(g_i)$ *hold for any* $i = 1,..., N$. *Then*

(i) $I\{h, \bar{g}, T, \bar{a}, \bar{b}\} = \sup_{1\le i\le N} h\circ g_i^{-1}(|a_i - b_i|)$,

and,

(ii) for any $\nu \ge I\{h, \bar{g}, T, \bar{a}, \bar{b}\}$ *there exist* $X_{\nu n} \in \mathcal{S}(T, \bar{g}, \bar{a})$ and $Y_{\nu n} \in \mathcal{S}(T, \bar{g}, \bar{b})$ *such that* $E\, h(\|X_{\nu n} - Y_{\nu n}\|) \to Y$ *as* $n \to \infty$.

Proof Claim 1 $I\{h, \bar{g}, T, \bar{a}, \bar{b}\} \ge \sup_{1\le i\le N} \varphi_i(a_i, b_i)$ *where* $\varphi_i(a_i, b_i) := h\circ g_i^{-1}(|a_i - b_i|)$.

Let $X, Y \in \mathcal{S}(C[0, 1])$. Then, as in claim 1 of theorem 1, by $A(h, g_i)$, $C(g_i)$ and the Jensen's inequality, we have

$$h^{-1}(Eh(\|X - Y\|)) \ge g_i^{-1}\, E\Big[g_i(|X(t_i) - Y(t_i)|)\Big]$$
$$\ge g_i^{-1}(|Eg_i(|X(t_i)|) - E\, g_i(|Y(t_i)|)|)$$
$$= h^{-1}\circ \varphi_i(a_i, b_i)$$

which proves the claim.

Claim 2 *For any* $\epsilon > 0$ *there exists a pair* $(X_\epsilon, Y_\epsilon) \in \mathcal{S}(T, \bar{g}, \bar{a}) \times \mathcal{S}(T, \bar{g}, \bar{b})$ *such that*

$$E\, h(\|X_\epsilon - Y_\epsilon\|) = \sup_{1\le i\le N} h \circ g_i^{-1}(|a_i - b_i| + \epsilon)\,\frac{|a_i - b_i|}{|a_i - b_i| + \epsilon}. \tag{4.12}$$

Without loss of generality we can assume that $a_i > b_i$, $i = 1,..., N$. Let $p_i := (a_i - b_i)/(a_i - b_i + \epsilon)$ and $q_i := 1 - p_i$, $i = 1,..., N$. We rearrange the indices i

so that $p_1 \le p_2 \le \dots \le p_N$. Choose sets $A_i \in \mathcal{A}$ such that $A_1 \subset \dots \subset A_N$ and $Pr(A_i) = p_i$ by using the assumption of $(\Omega, \mathcal{A}, Pr)$ being a nonatomic space. More precisely, since $(\Omega, \mathcal{A}, Pr)$ is nonatomic then for any $B \in \mathcal{A}$ and any $\lambda \in [0, Pr(B)]$ there exists $C = C(B, \lambda) \in \mathcal{A}$, $C \subseteq B$ such that $Pr(C) = \lambda$ (see Loeve (1977), p. 101). Then the required sets A_k $(k = 1,\dots, N)$ are given by

$$A_k = C(A_{k+1}, p_k),\ k = 1,\dots,N\ (A_{N+1} := \Omega).$$

Further, for any $\omega \in \Omega$, define

$$X_\epsilon(t_i, \omega) := \begin{cases} c_i := g_i^{-1}(a_i - b_i + \epsilon), & \text{if } \omega \in A_i \\ d_i := g_i^{-1}(b_i/q_i), & \text{if } \omega \notin A_i \end{cases}$$

and

$$Y_\epsilon(t_i, \omega) := \begin{cases} 0, & \text{if } \omega \in A_i \\ d_i, & \text{if } \omega \notin A_i . \end{cases}$$

We define $(t, X_\epsilon(t))_{t \in [0,1]}$ to be a random polygon line with vertices $(t_i, X_\epsilon(t_i))$ and letting $X_\epsilon(0, \omega) = 0$ if $t_1 > 0$ and $X_\epsilon(1, \omega) = 0$ if $t_N < 1$ for any $\omega \in \Omega$. Analogously define the process $Y_\epsilon(t)$. Then $E\, g_i(|X_\epsilon(t_i)|) = g_i(c_i)p_i + g_i(d_i)q_i = a_i$ and $E\, g_i(|Y_\epsilon(t_i)|) = g_i(d_i)\, q_i = b_i$ for any $i = 1,\dots, N$, i.e. $X_\epsilon \in \mathcal{S}(T, \bar{g}, \bar{a})$ and $Y_\epsilon \in \mathcal{S}(T, \bar{g}, \bar{b})$. Further,

$$(4.13) \qquad E\, h(\|X_\epsilon - Y_\epsilon\|) = E\, h\left[\sup_{t \in T} |X_\epsilon(t) - Y_\epsilon(t)|\right],$$

where

$$(4.14) \qquad |X_\epsilon(t_i, \omega) - Y_\epsilon(t_i, \omega)| = \begin{cases} c_i & \text{if } \omega \in A_i \\ 0 & \text{if } \omega \notin A_i \end{cases}$$

Since $g \in \mathcal{M}$ then for any $i = 1,..., N-1$ $p_i \le p_{i+1} \Leftrightarrow a_i - b_i \le a_{i+1} - b_{i+1} \Leftrightarrow c_i \le c_{i+1}$, i.e. $c_1 \le c_2 \le ... \le c_N$. Hence, by (4.14) and $A_1 \subset A_2 \subset ... \subset A_N$

$$(4.15) \qquad \sup_{t\in T} |X_\epsilon(t, \omega) - Y_\epsilon(t, \omega)| = \begin{cases} c_N & \text{if } \omega \in A_N, \\ 0 & \text{if } \omega \notin A_N. \end{cases}$$

Now, (4.13) and (4.15) imply that $E\, h(\|X_\epsilon - Y_\epsilon\|) = h(c_N)p_N =$ {right hand side of (4.12)} and thus the claim is proved.

Claim 1 and 2 prove the desired equality (i).

Claim 3 (ii) *is satisfied.*

Let $\tau \in (0,1)$, $\tau \notin T$. Using the same notations as in Claims 1 and 2 we define,

$X_\nu(t_i, \omega) := X_\epsilon(t_i, \omega)$, $Y_\nu(t_i, \omega) := Y_\epsilon(t_i, \omega)$, $\omega \in \Omega$,

$$X_\nu(\tau, \omega) = \begin{cases} h^{-1}(\nu) & \text{if } \omega \in A_N \\ 0 & \text{if } \omega \notin A_N \end{cases},$$

$Y_\nu(\tau, \omega) = 0$ for any $\omega \in \Omega$, and $\epsilon > 0$ is chosen so small that

$$\nu > h(c_N) := \sup_{1\le i\le N} h \circ g_i^{-1}(|a_i - b_i| + \epsilon).$$

We define the random broken lines X_ν and Y_ν with vertices $X_\nu(t_i)$, $Y_\nu(t_i)$, $i = 1,...,$ N and $X_\nu(0) = Y_\nu(0) = 0$ if $t_i > 0$, $X_\nu(1) = Y_\nu(1) = 0$ if $t_N < 1$. Hence, as in Claim 2 we conclude that

$$\|X_\nu(\cdot, \omega) - Y_\nu(\cdot, \omega)\| = \begin{cases} \max(c_N, h^{-1}(\nu)), & \text{if } \omega \in A_N \\ 0, & \text{if } \omega \notin A_N \end{cases}$$

Hence, $E\, h(\|X_\nu - Y_\nu\|) = \nu\, p_N = \nu \dfrac{a_N - b_N}{a_N - b_N + \epsilon}$. □

Theorem 4.3 *Let* $D(h, g_i)$ *hold for any* $i = 1,\ldots, N$. *Then*

$$I\{h, \bar{g}, T, \bar{a}, \bar{b}\} = 0 \tag{4.16}$$

and for any $\nu > 0$ *there exist random processes* $X_{\nu n} \in \mathcal{S}(T, \bar{g}, \bar{a})$ *and* $Y_{\nu n} \in \mathcal{S}(T, \bar{g}, \bar{b})$ *such that* $E\, h(\|X_{\nu n} - Y_{\nu n}\|) \to \nu$.

Proof. Claim 1. *For any* $n = 1,2,\ldots$ *there exist* $X_n \in \mathcal{S}(T, \bar{g}, \bar{a})$ *and* $Y_n \in \mathcal{S}(T, \bar{g}, \bar{b})$ *such that*

$$E\, h(\|X_n - Y_n\|) \le \sum_{i=1}^{N} \left(h(n\, b_i) \frac{b_i}{g_i(n\, b_i)} + h(n\, a_i) \frac{a_i}{g_i(n\, a_i)} \right). \tag{4.17}$$

Since $g_i \in \mathcal{M}$ for n large enough, say $n \ge n_0$ we can define disjoint sets A_{in}, B_{in}, C_{in} and such that $A_{in} + B_{in} + C_{in} = \Omega$ and

$$Pr(A_{in}) = c_{in} := \frac{a_i}{g_i(n\, a_i)}, \quad Pr(B_{in}) := d_{in} := \frac{b_i}{g_i(n\, b_i)}.$$

Now, for any $i = 1,\ldots,N$, $n \ge n_0$ define

$$\begin{cases} X_n(t_i, \omega) = n\, a_i\,, \quad Y_n(t_i, \omega) = 0 & \text{if } \omega \in A_{in}, \\ X_n(t_i, \omega) = 0\,, \quad Y_n(t_i, \omega) = n b_i & \text{if } \omega \in B_{in}, \\ X_n(t_i, \omega) = Y_n(t_i, \omega) = 0 & \text{if } \omega \in C_{in}. \end{cases} \tag{4.18}$$

Then $E\, g_i\left[|X_n(t_i)|\right] = g_i(n\, a_i)\, c_{in} = a_i$ and $E\, g_i\left[|Y_n(t_i)|\right] = g_i(n\, b_i)\, d_{in} = b_i$, i.e. $X \in \mathcal{S}(T, \bar{g}, \bar{a})$ and $Y \in \mathcal{S}(T, \bar{g}, \bar{b})$. Further, we define the random broken lines $X_n(t)$, $Y_n(t)$ $(t \in [0, 1])$ in the way we already did in Theorems 4.1 and 4.2. Without loss of generality we can assume that

$$a_1 \le a_2 \le \ldots \le a_N \le b_1 \le b_2 \le \ldots \le b_N .$$

Then

$$\|X_n - Y_n\| = \sup_{t \in T} |X_n(t) - Y_n(t)| \tag{4.19}$$

$$= \begin{cases} n\, b_N & \text{if } \omega \in B_{N,n}, \\ n\, b_{N-1} & \text{if } \omega \in B_{N-1,n} \setminus B_{N,n}, \\ \vdots & \\ n\, b_1 & \text{if } \omega \in B_{1,n} \setminus \bigcup_{j=2}^{N} B_{j,n}, \\ n\, a_N & \text{if } \omega \in A_{N,n} \setminus \bigcup_{j=1}^{N} B_{j,n}, \\ \vdots & \\ n\, a_1 & \text{if } \omega \in A_{1,n} \setminus \left[\bigcup_{j=2}^{N} A_{j,n} \cup \bigcup_{j=1}^{N} B_j\right], \\ 0 & \text{if } \omega \notin \bigcup_{j=1}^{N} A_{j,n} \cup \bigcup_{j=1}^{N} B_{j,n} . \end{cases}$$

Hence,

$$E\, h(\|X_n - Y_n\|) \le \sum_{j=1}^{N} h(n\, b_{j,n})\, d_{j,n} + \sum_{j=1}^{N} h(n\, a_{j,n})\, c_{j,n}$$

which proves (4.17) and the claim.

By $D(h, g_i)$ $(i = 1,\ldots,N)$ it follows that the right hand side of (4.17) goes to 0 as $n \to \infty$.

Hence, (4.16) holds true.

Claim 2. (ii) *is valid.*

Let $\tau \in (0,1)$, $\tau \notin T$ and define $\nu_n = \dfrac{\nu}{1 - \Pr(\bigcap_{j=1}^{N} C_{j,n})}$, $X_{\nu,n}(\tau) = h^{-1}(\nu_n)$, $Y_{\nu,n}(\tau) = 0$. We define the random broken line $X_{\nu,n}(t)$ with vertices $X_{\nu,n}(t_j) = X_n(t_j)$ (see (4.18)) and $X_{\nu,n}(\tau)$ (cf. Claim 2 of Theorem 4.1). Following the same notations as in Claim 1 we have

$$h(nb_i)\Pr(B_{i,n} \setminus \bigcup_{j=i+1}^{N} B_{j,n}) \le h(n\,b_i)\,d_{in} \to 0 \quad \text{as} \quad n \to \infty \tag{4.20}$$

and

$$h(n\,a_i)\Pr\Big[A_{i,n} \setminus \Big[\bigcup_{j=i+1}^{N} A_{j,n} \cup \bigcup_{j=1}^{N} B_{j,n}\Big]\Big] \le h(n\,a_i)\,c_{in} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.21}$$

Hence, by (4.19) – (4.21), for n large enough,

$$E\,h(\|X_{\nu,n} - Y_{\nu,n}\|) = \sum_{i=1}^{N} \max(\nu_n, h(nb_i))\Pr(B_{i,n} \setminus \bigcup_{j=i+1}^{N} B_{j,n})$$

$$+ \sum_{i=1}^{N} \max(\nu_n, h(na_i))\Pr(A_{i,n} \setminus [\bigcup_{j=i+1}^{N} A_{j,n} \cup \bigcup_{j=1}^{N} B_{j,n}])$$

$$= \sum_{i=1}^{N} \nu_n \Pr(B_{i,n} \setminus \bigcup_{j=i+1}^{N} B_{j,n})$$

$$+ \sum_{i=1}^{N} \nu_n \Pr(A_{i,n} \setminus [\bigcup_{j=i+1}^{N} A_{j,n} \cup \bigcup_{j=1}^{N} B_{j,n}])$$

$$= \nu_n \Pr\Big(\bigcup_{j=1}^{N} (A_{j,n} \cup B_{j,n})\Big) = \nu . \qquad \square$$

Theorem 4.4. *For any* $g_i \in M$ $(i = 1,..., N)$,

(i)

$$I\{0, \bar{g}, \bar{a}, \bar{b}\} := \inf\{\Pr(X \neq Y) : X \in \mathcal{S}(T, \bar{g}, \bar{a}),\ Y \in \mathcal{S}(T, \bar{g}, \bar{b})\} = 0 \tag{4.22}$$

and,

(ii) *for any* $\nu \in (0, 1)$ *there exists a sequence* $(X_{\nu n}, Y_{\nu n}) \in \mathcal{S}(T, \bar{g}, \bar{a}) \times \mathcal{S}(T, \bar{g}, \bar{b})$ *such that*

$$\Pr(X_{\nu n} \neq Y_{\nu n}) \to \nu \quad \text{as} \quad n \to \infty .$$

Proof. (i) Let $C_n \in \mathcal{A}$ and $\Pr(C_n) = \frac{1}{n}$. For any $i = 1,..., N$ define

$$\begin{cases} X_n(t_i, \omega) := g_i^{-1}(na_i)\ ,\ \ Y_n(t_i, \omega) := g_i^{-1}(nb_i)\ , \text{ if } \omega \in C_n\ , \\ X_n(t_i, \omega) := Y_n(t_i, \omega) = 0 \qquad , \qquad \text{if } \omega \notin C_n\ . \end{cases} \tag{4.23}$$

Then (4.23) determines the random polygon lines $X_n \in \mathcal{S}(T, \bar{g}, \bar{a})$ and $Y_n \in \mathcal{S}(T, \bar{g}, \bar{b})$. Since $X_n(t_i, \omega) = Y_n(t_i, \omega) = 0$ whenever $\omega \notin C_n$ and $i = 1,..., N$ then $X_n(t, \omega) = Y_n(t, \omega) = 0$ if $\omega \notin C_n$ and $t \in [0, 1]$. Hence,

$$\Pr(X_n = Y_n) \geq \Pr(\Omega \backslash C_n) = \frac{n-1}{n} \to 1$$

as desired.

(ii) Let $0 < \nu < 1$ and $\tau \in (0, 1)\backslash T$. Choose $A \in \mathcal{A}$ with $\Pr(A) = \nu$ and let

$$(4.24) \qquad \left\{ \begin{array}{l} X_{n\nu}(\tau, \omega) = 1 \ , \ Y_{n\nu}(\tau, \omega) = 0 \quad , \quad \text{for any } \omega \in A \\ X_{n\nu}(\tau, \omega) = Y_{n\nu}(\tau, \omega) = 0 \quad , \quad \text{for any } \omega \notin A \ , \\ X_{n\nu}(t_i) = X_n(t_i) \ , \ Y_{n\nu}(t_i) = Y_n(t_i) \ , \ i = 1, \ldots, N \ , \end{array} \right\}$$

where $X_n(t_i)$ and $Y_n(t_i)$ are given by (4.23). We construct the random broken lines $X_{n\nu}$ and $Y_{n\nu}$ by using (4.24) (cf. Claim 3 of Theorem 4.1). From the implications

$$X_{n\nu}(\cdot, \omega) = Y_{n\nu}(\cdot, \omega) \Leftrightarrow \left\{ \begin{array}{l} X_{n\nu}(t_i, \omega) = Y_{n\nu}(t_i, \omega), i = 1, \ldots, N \\ \text{and} \\ X_{n\nu}(\tau, \omega) = Y_{n\nu}(\tau, \omega) \end{array} \right\}$$

$$\Leftrightarrow \omega \notin A \cap C_n \ ,$$

it follows that

$$\Pr(X_{n\nu} = Y_{n\nu}) = \Pr(\Omega\backslash(A \cup C_n)) \to 1 - \nu$$

which proves (ii). □

As a consequence of Theorems 4.1–4.4 we obtain the following solution of problem 5 for $q_i(t) = t^{q_i} (q_i > 0)$ and $h(t) = t^p$ ($p \geq 0$, with the convention $t^0 := I\{t \neq 0\}$).

Corollary 4.1. *Let* $\bar{q} = (q_1, \ldots, q_N)$ $(q_i > 0)$, $p \geq 0$ *and*

$$I\{p, \bar{q}, \bar{a}, \bar{b}\} := \inf\{E \, \|X - Y\|^p, \ X, Y \in \mathcal{S}(C[0, 1]) \ , \ E \, |X(t_i)|^{q_i} = a_i \ ,$$

$$E|Y(t_i)|^{q_i} = b_i \ , \ i = 1, \ldots, N\} \ .$$

Then

$$(4.25)\quad I\{p, \bar{q}, \bar{a}, \bar{b}\} = \begin{cases} \sup\limits_{1\le i\le N} |a_i^{1/q_i} - b_i^{1/q_i}|^p, & \text{if } p \ge q_i \ge 1,\ i = 1,\dots,N, \\ \sup\limits_{1\le i\le N} |a_i - b_i|^{p/q_i}, & \text{if } p \ge q_i;\ 0 < q_i < 1,\ i = 1,\dots,N, \\ 0, & \text{if } 0 \le p \le q_i,\ i = 1,\dots,N \end{cases}$$

moreover for any $\nu > I\{p, \bar{q}, \bar{a}, \bar{b}\}$ *there exists a sequence* $(X_{n\nu}, Y_{n\nu}) \in \mathcal{S}(([0, 1]))$ *such that*

$$\mathbf{E}\,\|X_{n\nu} - Y_{n\nu}\|^p \to \nu \quad \text{as } n \to \infty$$

and

$$\mathbf{E}|X(t_i)|^{q_i} = a_i,\ \mathbf{E}|Y(t_i)|^{q_i} = b_i,\ i = 1,\dots,N.$$

Remark 4.2. Corollary 4.1 gives us an explicit expression for $I\{p, \bar{q}, \bar{a}, \bar{b}\}$ if p and q_i are subject to certain inequalities, (cf. (4.25)), or if $q_i = q$ for all $i = 1,\dots,N$. The problem of explicit description of $I\{p, \bar{q}, \bar{a}, \bar{b}\}$ for any $p \ge 0$ and $q_i > 0$ is still open.

5. Approximation of queueing systems with prescribed moments.

In this section the following two problems are subject to consideration: (a) Bounds for the deviation of output characteristics of two dependent queueing models;

(b) Approximation of queueing systems by deterministic type queueing models.

Consider the following problem which occurs in investigations stability of queueing models (see Kalashnikov and Rachev (1990), Chapter 5). Suppose two queueing models of type $G/G/1/\infty$, $(\bar{e}, \bar{s}, \bar{w})$ and $(\bar{e}^*, \bar{s}^*, \bar{w}^*)$, with dependent characteristics are given. Here $\bar{e} = \{e_n, n \in N\}$, $\bar{s} = \{s_n, n \in N\}$, $\bar{w} = \{w_n, n \in N\}$ are, respectively, the sequences of interarrival, service and waiting times. Assume that the components e_j, s_j, $j \in N$ of the "input flows" $\bar{e}$ and $\bar{s}$ are *dependent and non-identically distributed.*

The "output" flow is given by

$$w_1 = 0 \,,\; w_{n+1} = \max(0,\, w_n + s_n - e_n) \,. \tag{5.1}$$

Suppose also that the distribution of e_j (resp. s_j) is concentrated on a compact interval $[a_j, b_j]$ (resp. $[c_j, d_j]$). While this assumption is quite natural from the practical point of view, it is not used frequently in the literature, simply because it is easier to analyse queueing models with unbounded input distributions. We make similar assumptions for the model $(\bar{e}^*, \bar{s}^*, \bar{w}^*)$, in particular it is assumed that $a_j^* \le e_j^* \le b_j^*$, $c_j^* \le s_j^* \le d_j^*$ a.s. for all $j \in N$. The input pairs (e_j, e_j^*) , (s_j, s_j^*) of the two models are *arbitrarily mutually dependent,* the distributions of e_j's , e_j^*'s , s_j's , s_j^*'s are unknown. We assume that only the moments

$$\mathbf{E}\, e_j = \alpha_j,\; \mathbf{E}\, e_j^* = \alpha_j^*,\; \mathbf{E}\, s_j = \beta_j,\; \mathbf{E}\, s_j^* = \beta_j^* \tag{5.2}$$

are given. Our problem to find a sharp bound for the deviation between the waiting times in both models.

Let $\varphi_k(\bar{e}_{n-k,n-1},\, \bar{s}_{n-k,n-1})$ $(\bar{e}_{n-k,n-1} := (e_{n-k},\ldots,e_{n-1})$, $\bar{s}_{n-k,n-1} = (s_{n-k},\ldots,s_{n-1}))$ be the waiting time for the n-th arrival assuming that the system is

"free" at the moment n–k, in other words,

$$(5.3)\qquad \varphi_k(\bar{e}_{n-k,n-1}, \bar{s}_{n-k,n-1}) := \max\Big[0, e_{n-1} - s_{n-1}, (e_{n-1} - s_{n-1}) + (e_{n-2} - s_{n-2}), \ldots, (e_{n-1} - s_{n-1}) + (e_{n-2} - s_{n-2}) + \ldots + (e_{n-k} - s_{n-k})\Big] .$$

As a measure of deviation between the waiting times of both systems we shall use

$$\delta_p(T) := \sup_{n\in\mathbb{N}} \max_{1\le k\le T} \mathscr{L}_p\Big[\varphi_k(\bar{e}_{n,n+k-1}, \bar{s}_{n,n+k-1}), \varphi_k(\bar{e}^*_{n,n+k-1}, \bar{s}^*_{n,n+k-1})\Big] ,$$

where $p \ge 1$ and $T \ge 2$ are fixed, and

$$(5.4)\qquad \mathscr{L}_p(X,Y) := \{\mathbf{E}\,|X - Y|^p\}^{1/p} , \quad p \ge 1,\ X, Y \in \mathscr{S}(\mathbf{R}) .$$

For random vectors we extend (5.4) as follows

$$(5.5)\qquad \mathscr{L}_p(\bar{X}, \bar{Y}) = \{\mathbf{E}\|\bar{X} - \bar{Y}\|^p\}^{1/p} , \ \bar{X}, \bar{Y} \in \mathscr{S}(\mathbf{R}^T) ,$$

where $\|(x_1,\ldots,x_T)\| = \sum_{i=1}^{T} |x_j|$. Since φ_k is a Lipschitz function with respect to the Minkowski norm $\|\cdot\|$, we have that for any $k = 1,\ldots,T$.

$$(5.6)\qquad \mathscr{L}_p\Big[\varphi_k(\bar{e}_{n,n+k-1}, \bar{s}_{n,n+k-1}), \varphi_k(\bar{e}^*_{n,n+k-1}, \bar{s}^*_{n,n+k-1})\Big]$$

$$\le \mathscr{L}_p\Big[(\bar{e}_{n,n+T-1}, \bar{s}_{n,n+T-1}), (\bar{e}^*_{n,\,n+T-1}, \bar{s}_{n,n+T-1})\Big]$$

$$+ \mathscr{L}_p\left[(\bar{e}^*_{n,n+T-1}, \bar{s}_{n,n+T-1}), (\bar{e}^*_{n,n+T-1}, \bar{s}^*_{n,n+T-1})\right]$$

$$\leq \sum_{j=n}^{n+T-1} \left[\mathscr{L}_p(e_j, e^*_j) + \mathscr{L}_p(s_j, s^*_j)\right].$$

Now we invoke Theorem 2.1 to obtain sharp estimates of $\mathscr{L}_p(e_j, e^*_j)$ and $\mathscr{L}_p(s_j, s^*_j)$. Namely,

$$\mathscr{L}_p(e_j, e^*_j) \leq (D_j \delta_j + s_j)^{1/p}, \tag{5.7}$$

where

$$D_j := D_j(a_j, b_j, a^*_j, b^*_j) := |b_j - b^*_j|^p + |a_j - a^*_j|^p - |b_j - a^*_j|^p - |a_j - b^*_j|^p \tag{5.8}$$

$$T_j := T_j(a_j, a^*_j, b_j, b^*_j, \alpha_j, \alpha^*_j) := (1 - B_j)\,|b_j - a^*_j|^p$$

$$+ (B_j + C_j - 1)|a_j - a^*_j|^p + (1 - C_j)\,|a_j - b^*_j|^p, \tag{5.9}$$

$$B_j := \frac{b_j - \alpha_j}{b_j - a_j}, \quad C_j := \frac{b^*_j - \alpha^*_j}{b^*_j - a^*_j}, \tag{5.10}$$

$$\delta_j := \delta_j(a_j, a^*_j, b_j, b^*_j, \alpha_j, \alpha^*_j) := \max(0, 1 - B_j - C_j). \tag{5.11}$$

Remark 5.1 If e_j and e_j^* are unknown, i.e. $a_j = a_j^* = 0$, $b_j = b_j^* = +\infty$ then

$$\sup\{\mathscr{L}_p(e_j, e_j^*) : \mathbf{E}\, e_j = \alpha_j, \mathbf{E}\, e_j^* = \alpha_j^*\} = \infty$$

(cf. Kuznezova and Rachev (1987)).

In a similar way,

$$\mathscr{L}_p(s_j, s_j^*) \le (\hat{D}_j \hat{\delta}_j + \hat{T}_j)^{1/p}, \tag{5.12}$$

where $\hat{D}_j$, $\hat{\delta}_j$, $\hat{T}_j$ are defined by (5.8) – (5.11) changing b_j with d_j, b_j^* with d_j^*, a_j with c_j, and a_j^* with c_j^*. This lead us to the following

Theorem 5.1. For any $p \ge 1$ and $T = 2, 3, \ldots$

$$\delta_p(T) \le T \sup_{j \ge 1} [(D_j \delta_j + T_j)^{1/p} + (\hat{D}_j \hat{\delta}_j + \hat{T}_j)^{1/p}]. \tag{5.13}$$

The estimate is sharp or nearly sharp since the inequalities (5.7) and (5.12) are the best possible under the moment assumptions (5.2) (cf. Theorem 2.1), and also, the inequality (5.6) can not be improved in the set of all possible input flows $\bar{e}, \bar{e}^*, \bar{s}, \bar{s}^*$, .

Next we shall consider much more general case than the single channel models discussed above. Suppose the dynamics of a queueing system is determined by the transformation $\mathscr{T}$ from the set of $\mathscr{U}$ of input flows U to the set $\mathscr{V}$ of output flows V. Let V_0 represent the output at moment zero, V_0 is assumed to be an l-dimensional random vector, i.e. $V_0 \in \mathscr{S}(\mathbf{R}^l)$. It is quite general to assume that the

input and the output flows have the forms $U = (V_o, U_o, U_1, \ldots)$ and $V = (V_o, V_1, \ldots)$ where $U_j \in \mathcal{S}(\mathbf{R}^k)$. We endow $\mathcal{U}$ and $\mathcal{Y}$ with the norms

$$\|U\|_{\mathcal{U}} := \sum_{j=0}^{\infty} 2^{-j}\|U_j\|_{k,p} + \|V_o\|_{l,p}, \tag{5.14}$$

and

$$\|V\|_{\mathcal{Y}} := \sum_{j=0}^{\infty} 2^{-j}\|V_j\|_{l,p}, \tag{5.15}$$

where $p \geq 1$, $\|U_j\|_{k,p} := (\mathbf{E}\|U_j\|_k^p)^{1/p}$,

$\|U_j\|_k = \|(U_j^{(1)}, \ldots, U_j^{(k)})\| = |U_j^{(1)}| + \ldots + |U_j^{(k)}|$, and $\|V_j\|_{l,p}$ is defined in a similar way. Suppose the transformation $\mathcal{F}: \mathcal{U} \to \mathcal{Y}$ is determined by the set of mappings

$$\mathcal{F}_j : \mathbf{R}^l \times \mathbf{R}^{kj} \to \mathbf{R}^l, \; j \in \mathbf{N} \tag{5.16}$$

such that the output at "time" j is defined recursively:

$$V_j = \mathcal{F}_j(V_o, U_o, \ldots, U_{j-1}). \tag{5.17}$$

A smoothness assumption on $\mathcal{F}_j$ is given by the Lipschitz condition

$$\|F_j(\alpha_o, \beta_o, \ldots, \beta_{j-1})\|_l \leq c_j[\|\alpha_o\|_l + \sum_{i=0}^{j-1} \|\beta_j\|_k]. \tag{5.18}$$

A reasonably large number of queueing models meet the condition (5.16) – (5.18). Among them are the single–channel models $G|G|1|\infty$, the multichannel models $G|G|J|\infty$, the multichannel–multiphased model $(G|G|J_1) \to (G|J_2) \to \to \ldots \to (G|J_n)$ and others (cf. Kalashnikov and Rachev (1990), Chapter 5). By (5.15), (5.17) and (5.18), $\|V_j\|_{l,p} \le c_j[\|V_0\|_{l,p} + \sum_{i=0}^{j-1} \|U_i\|_{k,p}]$ and thus

$$\|V\|_{\mathcal{V}} \le 2c_j \|U\|_{\mathcal{U}} \le 2c_j[\sum_{i=1}^{l} \|V_0^{(i)}\|_{l,p} + \sum_{i=1}^{k} \sum_{j=0}^{\infty} 2^{-j} \|V_j^{(i)}\|_{l,p}]. \tag{5.19}$$

Combining (5.19) with Theorem 2.1 gives us a sharp bound of the deviation of two queueing models $V = \mathcal{F}U$, $V^* = \mathcal{F}U^*$ which dynamics is determined by (5.14) – (5.18).

Theorem 5.2. *Suppose*

$$V = \mathcal{F}U,\ U \in \mathcal{U},\ V \in \mathcal{V} \tag{5.20}$$

is a queueing model satisfying (5.14) – (5.18) *such that*

$$a_0^{(i)} \le V_0^{(i)} \le b_0^{(i)} \text{ a.s., } \mathbf{E}\, V_0^{(i)} = \alpha_0^{(i)},\ i = 1,\ldots, l$$

$$c_j^{(i)} \le V_j^{(i)} \le d_j^{(i)} \text{ a.s., } \mathbf{E}\, V_j^{(i)} = \beta_j^{(i)} j = 0,1,\ldots,\ i = 1,\ldots, k\,.$$

Alongside the model (5.20) *considers the same type model indexed by* $*$ *and satisfying the above two sets of inequalities with constants indexed by* $*$. *Then*

$$\|V - V^*\|_{\gamma} \le 2c_j\Big[\sum_{i=1}^{l} (D_0^{(i)}\,\delta_0^{(i)} + T_0^{(i)})^{1/p} + \sum_{i=1}^{k}\sum_{j=0}^{\infty} 2^{-j}(\tilde{D}_j^{(i)}\,\tilde{\delta}_j^{(i)} + \tilde{T}_j^{(i)})^{1/p}\Big],$$

where the D*'s,* δ*'s and* T*'s are determined by the same formulae as in* (5.8) – (5.10), *and*

$$D_0^{(i)} = D_i(a_0^{(i)}, b_0^{(i)}, a_0^{(i)*}, b_0^{(i)*}),$$

$$T_0^{(i)} = T_i(a_0^{(i)}, b_0^{(i)}, a_0^{(i)*}, b_0^{(i)*}, \alpha_0^{(i)}, \alpha_0^{(i)*}),$$

$$\delta_0^{(i)} = \delta_i(a_0^{(i)}, b_0^{(i)}, a_0^{(i)*}, b_0^{(i)*}, \alpha_0^{(i)}, \alpha_0^{(i)*}),$$

$$\tilde{D}_j^{(i)} = D_j(c_j^{(i)}, d_j^{(i)}, c_j^{(i)*}, d_j^{(i)*})$$

$$\tilde{T}_j^{(i)} = T_j(c_j^{(i)}, d_j^{(i)}, c_j^{(i)*}, d_j^{(i)*}, \beta_j^{(i)}, \beta_j^{(i)*}),$$

$$\tilde{\delta}_j^{(i)} = \delta_j(c_j^{(i)}, d_j^{(i)}, c_j^{(i)*}, d_j^{(i)*}, \beta_j^{(i)}, \beta_j^{(i)*}). \qquad \square$$

The rest of this section deals with Problem (b). Suppose again that the "real" queueing system is determined by the triplet $(\bar{e}, \bar{s}, \bar{w})$ where $\bar{w}$ is given by the recursive equation (5.1). Often in practice one models the random input characteristics by replacing their random values with constants, usually equal to the corresponding means. In doing so, it is natural to investigate the deviation between the "real" output $\bar{w}$ and the modelled ("ideal") output $\bar{w}^*$. (In the sequel, all quantities related to the approximating model will have the same notations as in the "real" system but superscribed with *.)

The deviation between $\bar{w}$ and $\bar{w}^*$ will be expressed by the *generalized Kantorovich metric* l_p defined as follows: for $X, Y \in \mathcal{S}(\mathbf{R}^\infty)$,

$$(5.21) \qquad l_p(X, Y) := l_p(\mathrm{Pr}_X, \mathrm{Pr}_Y) := \min\{\mathcal{L}_p(\tilde{X}, \tilde{Y}) : \tilde{X}, \tilde{Y} \in \mathcal{S}(\mathbf{R}^\infty),\ \tilde{X} \overset{d}{=} X,\ \tilde{Y} \overset{d}{=} Y\},\ p > 0,$$

where $\mathcal{L}_p(\tilde{X}, \tilde{Y}) := [\mathbf{E}\, d^p(\tilde{X}, \tilde{Y})]^q$, $q = \min(1, 1/p)$. In the above definition, the space $\mathcal{S}(\mathbf{R}^\infty)$ consists of all random sequences taking values in the metric space $(\mathbf{R}^\infty, d)$, where $d(\bar{x}, \bar{y}) := \sum_{j=1}^{\infty} 2^{-j}\|x_i - y_i\|$. Since we have assumed that the underlined probability space is not atomic, the minimum in the RHS of (5.21) is equal to

$$\min\Big\{\int_{\mathbf{R}^\infty \times \mathbf{R}^\infty} d(x, y)\, P(dx, dy)\ ;\ P\text{'s are probabilities on } \mathbf{R}^\infty \times \mathbf{R}^\infty \text{ with fixed projections } \mathrm{Pr}_X \text{ and } \mathrm{Pr}_Y\Big\}.$$

In the space $\mathcal{S}_p(\mathbf{R}^\infty)$ of random sequences X with components having finite pth absolute moment, $l_p(X_n, X) \to 0$ for $X_n, X \in \mathcal{S}_p(\mathbf{R}^\infty)$ if and only if X_n converges to X in distribution $(X_n \overset{d}{\Rightarrow} X)$ and $\mathbf{E}\, d^p(X_n, a) \to \mathbf{E}\, d^p(X, a)$ for some (and thus for any) fixed point $a \in (\mathbf{R}^\infty, d)$. This assertion follows from Rachev (1989), Theorem 2, and since $\mathcal{S}_p(\mathbf{R}^\infty) \equiv \{X \in \mathcal{S}(\mathbf{R}^\infty),\ \mathbf{E}\, d^p(X, a) < \infty\}$. In the space $\mathcal{S}_p(R)$ of real-valued random variables with finite pth absolute moment, l_p is defined again by (5.21) with $\mathcal{L}_p(X, Y) = \{\mathbf{E}\,|X - Y|^p\}^q$. For $p \geq 1$, $X, Y \in \mathcal{S}_p(R)$

$$l_p(X, Y) = \Big[\int_0^1 |F_X^{inv}(t) - F_Y^{inv}(t)|^p dt\Big]^{1/p}$$

where F_X^{inv} is the generalized inverse of the distribution function F_X, see for example Theorem 1, Chapter 3, p.7 in Kalashnikov and Rachev (1990).

For $\bar{X}^{(n)} = (X_1^{(n)}, X_2^{(n)}, \ldots) \in \mathcal{S}_p(R^\infty)$, $\bar{X} = (X_1, X_2, \ldots) \in \mathcal{S}_p(R^\infty)$, $l_p(\bar{X}^{(n)}, \bar{X}) \geq 2^{-j} l_p(X_j^{(n)}, X_j)$

and thus $l_p(\bar{X}^{(n)}, \bar{X}) \to 0$ implies the weak convergence of any j–component $X_j^{(n)} \xrightarrow{d} X_j$ and $\mathbf{E}\,|X_j^{(n)}|^p \to \mathbf{E}\,|X_j|^p$. The convergence of the pth moments makes the use of the generalized Kantorovich metric l_p more appropriate than the Prokhorov metric (cf. Anastassiou and Rachev (1990)).

Further we consider two types of approximating queues $D|G|1|\infty$ (i.e. e_j^* are constants) and $D|D|1|\infty$ (i.e. e_j^* and s_j^* are constants). Similar results can be obtained if one examines the model $G|D|1|\infty$ (i.e. s_j^* are constants) as an approximation of the "real" queue $G|G|1|\infty$.

In both queues $D|G|1|\infty$ and $G|G|1|\infty$ the sequences of service times $\bar{s}^*$ and $\bar{s}$ consist of dependent nonidentically distributed random variables. The next lemma shows that the outputs for the ideal and real models meet a lower bound of deviation if $\bar{s}^*$ is chosen to have independent components.

Let $\epsilon > 0$ and $X = (X_1, X_2, \ldots) \in \mathcal{S}_p(R^\infty)$, the components of X are said to be (l_p, ϵ)–independent if

$$\text{(5.22)} \qquad \text{IND}(X) := l_p(X, \underline{X}) \leq \epsilon,$$

where $\underline{X}_i$'s (the components of $\underline{X}$) are independent and $\underline{X}_i \overset{d}{=} X_i$ $(i \in N)$.

Lemma 5.1. *Let the approximating model be of type* $D|G|1|\infty$. *Assume that the sequences* $\bar{e}$ *and* $\bar{s}$ *of the queueing model* $G|G|1|\infty$ *are independent. Then*

$$\text{(5.23)} \qquad \tfrac{1}{2} l_p(\bar{w}, \bar{w}^*) \leq \text{IND}(\bar{s}) + \text{IND}(\bar{s}^*) + \sum_{j=1}^{\infty} 2^{-j} (l_p(e_j, e_j^*) + l_p(s_j, s_j^*)).$$

Proof. Using the recursive equations (5.1) for $\bar{w}$ and $\bar{w}^*$, one obtains $\frac{1}{2} d(\bar{w}, \bar{w}^*) \le d(\bar{e}, \bar{e}^*) + d(\bar{s}, \bar{s}^*)$ (cf. Anastassiou and Rachev (1990), Theorem 2.1). Hence, $\frac{1}{2} \mathscr{L}_p(\bar{w}, \bar{w}^*) \le \mathscr{L}_p(\bar{e}, \bar{e}^*) + \mathscr{L}_p(\bar{s}, \bar{s}^*)$. Since $\bar{e}$ and $\bar{s}$ (resp. $\bar{e}^*$ and $\bar{s}^*$) are independent then passing to the minimal metrics one gets

$$(5.24) \qquad \frac{1}{2} l_p(\bar{w}, \bar{w}^*) \le l_p(\bar{e}, \bar{e}^*) + l_p(\bar{s}, \bar{s}^*) .$$

By (5.21) and e_j = constant $(j \in n)$ we have

$$(5.25) \qquad l_p(\bar{e}, \bar{e}^*) \le \sum_{j=1}^{\infty} 2^{-j} \mathscr{L}_p(e_j, e_j^*) = \sum_{j=1}^{\infty} 2^{-j} l_p(e_j, e_j^*) .$$

To estimate $l_p(\bar{s}, \bar{s}^*)$ in the RHS of (5.24) we invoke the (l_p, ϵ)–independence (5.22):

$$(5.26) \qquad l_p(\bar{s}, \bar{s}^*) \le \mathrm{IND}(\bar{s}) + \mathrm{IND}(\bar{s}^*) + l_p(\underline{\bar{s}}, \underline{\bar{s}}^*) ,$$

where $\underline{\bar{s}}$ (resp. $\underline{\bar{s}}^*$) consists of independent components and $\underline{s}_j \overset{d}{=} s_j$ (resp. $\underline{s}_j^* \overset{d}{=} s_j^*$). We now need the "regularity" property of the generalized Kantorovich metric:

$$(5.27) \qquad l_p(X + Z, Y + Z) \le l_p(X, Y)$$

for any Z independent of X and Y in $\mathscr{X}_p(\mathbb{R}^\infty)$. In fact, for any $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathscr{X}_p(\mathbb{R}^\infty)$

$$(5.28) \qquad \mathscr{L}_p(\tilde{X} + \tilde{Z}, \tilde{Y} + \tilde{Z}) = \mathscr{L}_p(\tilde{X}, \tilde{Y})$$

and thus, passing to minimal metrics in (5.28) we obtain (5.27). Using the triangle inequality and (5.27) we have

$$(5.29) \qquad l_p\Big(\sum_{n=1}^{\infty} X^{(n)}, \sum_{n=1}^{\infty} Y^{(n)}\Big) \le \sum_{n=1}^{\infty} l_p(X^{(n)}, Y^{(n)})$$

for sequences $\{X^{(n)}\}_{n\ge 1} \subset \mathcal{X}_p(R^\infty)$, $\{Y^{(n)}\}_{n\ge 1} \subset \mathcal{X}_p(R^\infty)$ of independent components. Let E^j be a sequence with components all equal to zero except the jth component which equals 1. Then by (5.29)

$$(5.30) \qquad l_p(\bar{s}, \bar{s}^*) = l_p\Big(\sum_{j=1}^{\infty} \bar{s}\, E^j, \sum_{j=1}^{\infty} \bar{s}^*\, E^j\Big)$$
$$\le \sum_{j=1}^{\infty} l_p(\bar{s}\, E^j, \bar{s}^*\, E^j) = \sum_{j=1}^{\infty} 2^{-j}\, l_p(s_j, s_j^*)$$

Combination of (5.24), (5.25), (5.26) and (5.30) proves the lemma. □

The bound (5.23) suggests that the approximating model should be chosen with $\bar{s}^*$ having independent components. If this is the case then $IND(\bar{s}^*) = 0$ and the first problem is to estimate $IND(\bar{s})$.

Lemma 5.2 (a) *Suppose the only information known about the "real" service times are the moments*

$$(5.31) \qquad E\, s_j^{q_1} = \beta_j, \quad E\, s_j^{q_2} = \gamma_j, \quad j \in N$$

Suppose the support of F_{s_j} *is* $[0, \infty)$. *Then*

$$\text{(5.32)} \qquad \mathrm{IND}(\bar{s}) \le \sum_{j=1}^{\infty} 2^{-j} \Delta_j ,$$

where

$$\text{(5.33)} \qquad \Delta_j := \begin{cases} \left[2\,\beta_j^{1/q_1}\right]^{pq} , & \text{if } 0 < p \le q_1,\ 1 \le q_1 < q_2 , \\ +\infty , & \text{if } 0 < q_1 < q_2 < p \text{ and} \\ & \beta_j^{1/q_1} = \gamma_j^{1/q_2} . \end{cases}$$

and $q = \min(1, 1/p)$.

(b) *Suppose the support of* F_{s_j} *is the compact interval* $[c_j, d_j]$ *and* $\beta_j = E\, s_j$.

Then (5.32) *holds with*

$$\text{(5.34)} \qquad \Delta_j = (\tilde{D}_j\, \tilde{\delta}_j + \tilde{T}_j)^{1/p} , \ p \ge 1 , \ \textit{where}\ \tilde{D}_j = -2(d_j - c_j)^p ,$$

$$\tilde{T} = 2\left(1 - \frac{d_j - \beta_j}{d_j - c_j}\right) , \ \textit{and}\ \tilde{\delta}_j = \max\left(0, 1 - 2\,\frac{d_j - \beta_j}{d_j - c_j}\right) .$$

Indication: (a) follows from Corollary 2 of Kuznezova and Rachev (1987), and (b) from Theorem 2.1.

Lemma 5.3. *Suppose for any* $j \in \mathbf{N}$ *the first two moments of* e_j *are known:*

(5.35) $\quad m_j := \mathbf{E}\, e_j$, $m_j^{(2)} := \mathbf{E}\, e_j^2$, $\sigma_j^2 := \mathrm{Var}\, e_j$. *Let* $a_j \le e_j \le b_j$ *a.s.*

(i) *Suppose* $p \ge 2$ *and* $-\infty < a_j < b_j < \infty$. *Then if* e_j^* *is chosen to be the midpoint of* $[a_j, b_j]$, *then*

$$(5.36) \qquad l_p(e_j, e_j^*) \le \left[\left[\frac{b_j - a_j}{2}\right]^{p-2} \left[m_j^{(2)} - m_j(a_j + b_j) + e_j^{*2}\right]\right]^{1/p}.$$

(ii) *Suppose* $0 < p \le 2$ *and either* $-\infty = a_j$, $+\infty = b_j$, or $-\infty < a_j < b_j < \infty$ *and*

$$(5.37) \qquad \sigma_j \le \min[m_j - a_j, b_j - m_j].$$

Then the "optimal" e_j^* *for the approximating model is given by* $e_j^* = m_j$ *and in this case*

$$(5.38) \qquad l_p(e_j, e_j^*) \le \sigma_j^{pq}, \; q = \min(1, 1/p).$$

Indication. This follows from Theorems 2.5, 2.6 and 2.7 after some obvious arguments. The estimates (5.36) and (5.38) are sharp. -

Remark 5.2. Condition (5.37) is fulfilled if for example e_j is uniformly distributed on $[a_j, b_j]$. But it is not fulfilled if for example, $\Pr(e_j = 0) = \frac{1}{4} = \Pr(e_j = a)$, $\Pr(e_j = \frac{a}{2}) = \frac{1}{2}$ and a is big enough.

Lemma 5.4. (a) *Suppose* $0 < p \le q_1$, $1 \le q_1 < q_2$. *Then*

(5.39) $$\sup\{l_p(s_j, s_j^*) : n_j = E\, s_j^{q_1},\ n_j^{(2)} = E\, s_j^{q_2},\ n_j^* = E\, s_j^{*q_1},\ n_j^{*(2)} = E\, s_j^{*q_2}\}$$

$$= (n_j^{1/q_1} + n_j^{*1/q_1})^{pq},\ q = \min(1, 1/p).$$

(b) *Suppose* $p \geq 1$, $c_j \leq s_j \leq d_j$, $c_j^* \leq s_j^* \leq d_j^*$ a.s., *and* $n_j = E\, s_j$, $n_j^* = E\, s_j^*$. *Then*

(5.40) $$l_p(s_j, s_j^*) \leq (\tilde{D}_j\, \tilde{\delta}_j + \tilde{T}_j)^{1/p} \quad \text{where } \tilde{D}_j = D_j(c_j, d_j, c_j^*, d_j^*),$$

$$\tilde{\delta}_j = \delta_j(c_j, d_j, c_j^*, d_j^*, n_j, n_j^*),\ \tilde{T}_j = T_j(c_j, d_j, c_j^*, d_j^*, n_j, n_j^*),$$

see (5.8) – (5.10).

Indication. (a) follows from Corollary 2 of Kuznezova and Rachev (1987) and (b) from Theorem 2.1. ∎

Lemmas 5.1 – 5.4 lead us to the main result.

Theorem 5.3. *Let the approximating queueing model be of type* $D|G|1|\infty$. *Assume that the sequences* $\bar{e}$ *and* $\bar{s}$ *of the "real" queueing model are independent. Then the generalized Kantorovich metric between the sequences of waiting times of the "approximating" and "real" models is bounded as follows*

$$l_p(\bar{w}, \bar{w}^*) \leq 2\, \mathrm{IND}(\bar{s}) + 2\, \mathrm{IND}(\bar{s}^*)$$

(5.41) $$+ \sum_{j=1}^{\infty} 2^{-j+1}(l_p(e_j, e_j^*) + l_p(s_j, s_j^*)).$$

Each term in the RHS of (5.41) can be estimated as follows:

(a) *an appropriate choice for the approximating sequence service times will be* $\mathrm{IND}(s^*) = 0$,

(b) *if* (5.31) *holds, a bound for* $\mathrm{IND}(\bar{s})$ *is given by* (5.32);

(c) *if the means and variances of* e_j's *are known then* $l_p(e_j, e_j^*)$ *can be estimated from above by* (5.36);

(d) *the last term in* (5.40), $l_p(s_j, s_j^*)$ *can be estimated by* (5.39) (*resp.* (5.40)) *provided the corresponding moment conditions hold.* ∎

In the next theorem we shall omit the restriction that $\bar{e}$ and $\bar{s}$ are independent but we shall assume that the approximating model is of completely deterministic type $D|D|1|\infty$.

Theorem 5.4. *Under the above assumption*

$$(5.42) \qquad l_p(\bar{w}, \bar{w}^*) \leqq \sum_{j=1}^{\infty} 2^{-j+1} (l_p(e_j, e_j^*) + l_p(s_j, s_j^*))$$

If the first moments of e_j *and* s_j *are fixed* (see (5.35)) *then* $l_p(e_j, e_j^*)$ (*resp.* $l_p(s_j, s_j^*)$) *can be estimated invoking Lemma 5.3.*

The proof is similar to the previous theorem. ∎

6. Rounding random numbers with fixed moments.

Suppose $\{X_i\}_{i \in \mathbf{N}}$ are nonnegative i.i.d. r.v.'s with known moments

$$(6.1) \qquad \mathrm{E}\, X_1 = d_1 \,, \; \mathrm{E}\, X_1^r = d_r \,.$$

The c-rounding $[\cdot]_c$ ($c \in [0, 1]$) gives us the sequence of i.i.d. roundings $\{[X_i]_c\}_{i \in \mathbf{N}}$. Let $V_i := [X_i]_c - X_i$ be *the rounding error* and $S_{n,c} = \sum_{i=1}^{n} V_{i,c}$ is the *total rounding error*. Then the *normalized rounding error* $n^{-1} S_{n,c}$ converges by LLN to $E\,[X_1]_c - d_1$. The objective of this section is to find sharp bounds for the distribution function of $n^{-1} S_{n,c}$ subject to (6.1).

In other words, for a suitably chosen metric μ in the distribution functions space, the problem is to determine the "radius" of the set of probabilistic laws, $\mathcal{L}(n^{-1} S_{n,c})$, i.e.

$$\mathcal{D}_n := \sup\{\mu(n^{-1} S_{n,c}, E\,[X]_c - d_1) : E\,X = d_1, E\,X^r = d_r\}. \tag{6.2}$$

In (6.2), X has the same distribution as X_i's, and thus $E[X]_c - d_1 = E\,V$, where $V \overset{d}{=} V_1$.

Clearly there is a great variety of metrics μ one can choose in (6.2). We shall consider two metrics, one especially designed for the problem, the so-called *"ideal metric of order* s" $\theta_s (s > 1)$ and the other is a representative of the so-called *"traditional"* metric, the well-known Levy metric L.

Note that Theorems 3.5 – 3.8 provide us with sharp bounds for $E\,[X]_c - d_1$. In fact,

$$\sup\{E\,[X]_c - d_1 : E\,X = d_1, E\,X^r = d_r\} = U - d_1, \tag{6.3}$$

$$\inf\{E\,[X]_c - d_1 : E\,X = d_1, E\,X^r = d_r\} = L - d_1, \tag{6.4}$$

where the exact values of U and L are given in Theorem 3.5 – 3.8. For the special

cases $c = 0$, $c = 1$, the above bounds are determined by Theorems 3.9 – 3.17. Next we can choose μ in the definition of $\mathscr{D}_n = \mathscr{D}_n(\mu)$ to be the Lévy metric L

$$L(X, Y) = \inf\{\epsilon > 0 : F_X(x - \epsilon) - \epsilon \le F_Y(x) \le F_X(x + \epsilon) + \epsilon \text{ for all } x \in R\},$$

and thus for the distribution function F_n of $n^{-1} S_{n,c}$ we obtain the following bounds

(6.5a) $\quad 0 \le F_n(x) \le \mathscr{D}_n$ for $0 \le x \le L - \mathscr{D}_n$,

(6.5b) $\quad 0 \le F_n(x) \le 1$ for $L - \mathscr{D}_n \le x \le U + \mathscr{D}_n$,

(6.5c) $\quad 1 - \mathscr{D}_n \le F_n(x) \le 1$ for $U + \mathscr{D}_n \le x$.

From Theorems 3.5 – 3.17 it follows that the above bounds are sharp.

Our next step is to find a good estimate for $\mathscr{D}_n = \mathscr{D}_n(L)$. To this end we first estimate $\mathscr{D}_n(\theta_s)$ for

(6.5) $\quad \theta_s(X, Y) = \sup\{|E\, f(X) - E\, f(Y)|:$ f are bounded functions on R, with q–integrable second derivative, $\int |f''|^q \le 1\}$, $1 < s < 2$, $q = \frac{1}{2-s}$.

The next lemma shows that the θ_s – radius $\mathscr{D}_n(\theta_s) = O(n^{1-s})$ for *any* $1 < s < 2$.

Lemma 6.1. For $1 < s < 2$,

$$\mathscr{D}_n(\theta_s) \le \tilde{c}\, n^{1-s}, \quad \tilde{c} := \frac{2}{s}(c\, v(1 - c))^s$$

where $v := \max$.

Proof. For any X and Y with equal means

$$\theta_s(X,Y) \le \frac{1}{s}\kappa_s(X,Y)$$

$$:= \inf\{E|\tilde{X}|\tilde{X}|^{s-1} - \tilde{Y}|\tilde{Y}|^{s-1}| : \tilde{X} \overset{d}{=} X, \tilde{Y} \overset{d}{=} Y\}$$

(cf. Maejima and Rachev (1987)). Therefore, from the "ideality" of θ_s (see Zolatarev (1986), Chapter 1, Kalashnikov and Rachev (1990) Chapter 3),

$$\mathscr{D}_n(\theta_s) \le n^{1-s}\,\theta_s(V, EV)$$
$$\le n^{1-s}\,\frac{1}{s}\,\kappa_s(V, EV)$$
$$= n^{1-s}\,\frac{1}{s}\,E\,|\,V|V|^{s-1} - (EV)\,|EV|^{s-1}|$$
$$\le n^{1-s}\,\frac{1}{s},$$

the latter since $|V| = |X - [x]_c| \in (0, c\,v(1-c))$ □

In the next theorem we bound $\mathscr{D}_n = \mathscr{D}_n(L)$ in (6.2) as $O\left[n^{-\frac{s-1}{1+s}}\right]$.

Theorem 6.1. For any $1 < s < 2$, $0 < c < 1$,

$$\mathscr{D}_n(L) \le (4\,\tilde{c})^{\frac{1}{1+s}}\,n^{\frac{1-s}{1+s}},$$

where the constant $\tilde{c}$ is defined as in Lemma 6.1.

Proof. The following claim was proved by Grigorevski and Shiganov (1970) for the case $s = 2$, i.e. in (6.5) the functions f have a.e. f'' and $|f''| \le 1$ a.e.

Claim. For any $1 < s < 2$

$$\theta_s(X, Y) \ge \frac{1}{4}\,L^{1+s}(X,Y),$$

Proof. Let $L(X,Y) > \epsilon$. Then there exists x_0 such that either $F_X(x_0) > F_Y(x_0 + \epsilon) + \epsilon$ or $F_Y(x_0) > F_X(x_0 + \epsilon) + \epsilon$, say the first inequality takes place

Define $f_0(x) := 1$ for $x \le x_0$; $f_0(x) = 1 - \left[\frac{2(x - x_0)}{\epsilon}\right]^2$ for $x_0 < x \le x_0 + \frac{\epsilon}{2}$; $f_0(x) = -1 + \left[\frac{2(x_0 + \epsilon - x)}{\epsilon}\right]^2$ for $x_0 + \frac{\epsilon}{2} \le x < x_0 + \epsilon$; and $f_0(x) = 1$ for $x \ge x_0 + \epsilon$.

Observe that $|f_0(x)| \le 1$, $f_0''(x)$ exists a.e. and

$$\|f_0''\|_q = \left[\int_{x_0}^{x_0+\epsilon} |f_0''(x)|^q dx\right]^{1/q}$$

$$= 8\,\epsilon^{-s} =: c(\epsilon)$$

for $q = \frac{1}{2-s}$. Recalling the definition of θ_s we have

$$\theta_s(X,Y) \ge \left|E\left[\frac{f_0(X)}{c(\epsilon)}\right] - E\left[\frac{f_0(Y)}{c(\epsilon)}\right]\right|$$

$$= \frac{1}{c(\epsilon)} \left|\int (f_0(x) + 1) d\left[F_X(x) - F_Y(x)\right]\right|$$

$$= \frac{1}{c(\epsilon)} \left|\int_{-\infty}^{x_0} (f_0(x) + 1)\, dF_X(x) + \int_{x_0}^{\infty} (f_0(x) + 1)\, dF_X(x)\right.$$

$$\left. - \int_{-\infty}^{x_0+\epsilon} (f_0(x) + 1)\, dF_Y(x) - \int_{x_0+\epsilon}^{\infty} (f_0(x) + 1)\, dF_Y(x)\right|$$

$$\ge \frac{1}{c(\epsilon)} \left[\int_{-\infty}^{x_0} (f_0(x) + 1)\, dF_X(x) - \int_{x_0+\epsilon}^{\infty} (f_0(x) + 1)\, dF_Y(x)\right]$$

$$\ge \frac{2}{c(\epsilon)} \left[F_X(x_0) - F_Y(x_0 + \epsilon)\right]$$

$$\ge \frac{2\epsilon}{c(\epsilon)} = \frac{1}{4}\,\epsilon^{1+s}.$$

Letting $\epsilon \to L(X,Y)$ completes the proof of the claim.

Now the desired estimate follows from Lemma 6.1 and the claim. □

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